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# SOLUTIONS FOR CERTAIN RECTANGULAR SLABS CONTINUOUS OVER FLEXIBLE SUPPORTS

A REPORT OF AN INVESTIGATION

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THE ENGINEERING EXPERIMENT STATION
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IN COÖPERATION WITH

THE UNITED STATES BUREAU OF PUBLIC ROADS

AND

THE ILLINOIS DIVISION OF HIGHWAYS

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# SOLUTIONS FOR CERTAIN RECTANGULAR SLABS CONTINUOUS OVER FLEXIBLE SUPPORTS

## I. Introduction

1. *Introductory Remarks.*—In the design of bridge slabs subjected to wheel loads, certain questions have remained unanswered or only partially answered in spite of the considerable progress that has been made during the past decade. These questions pertain to the effect of continuity of the slab across stringers or floor beams, to the effect of curbs and diaphragms, to the effect of flexibility of the supporting beams, to the sufficiency of design on the basis of moment only, and to numerous other elements of undoubted importance in the proper design of slabs. To provide further information relative to some of these problems, a coöperative investigation was begun in the Engineering Experiment Station in September, 1936. The analytical part of this investigation included a reconnaissance of the methods of solution available for attacking the various problems. The present bulletin embodies the results obtained by the application of one of the methods, namely, the classical procedure of obtaining a solution of Lagrange's differential equation of the deflected middle surface of the slab, satisfying, at the same time, all of the boundary conditions at the various edges or interior supports.

A study of the use of an analytical procedure of this kind must be undertaken in the sense of analytical experimentation. The procedure virtually guarantees results in the form of equations, but the usableness of the equations is by no means assured. It will be seen that some of the results are of immediate practical value and others appear to be of purely academic interest. Some of the equations must necessarily be extended or modified to apply to types of structures currently being built. Others, by their very cumbersomeness, indicate that other methods of analysis should be applied to those particular problems.

Although the formulas are not suitable for direct use in design, a few illustrative curves and tables are given. It is contemplated that more of the equations will later be translated into numerical values and that the theoretical results will be interpreted in the light of test data. Certain formulas are capable of furnishing a means of evaluating, at least partially, the effect of continuity across the stringers or floor beams, the effect of flexibility of the supporting members, and the effect of diaphragms which are connected to the stringers but which are not themselves in contact with the slab.

As in most analytical treatments of structural problems, simplifying assumptions are made. For example, in addition to the usual assumptions of the ordinary slab theory, it is assumed that the supporting beams exert only vertical forces upon the slab. It is recognized that this assumption is not fulfilled in an actual bridge since bond and friction produce the effect of a T beam. There is consequently a shifting of the neutral surface of the slab in the neighborhood of the supports. Nevertheless, it is possible that a modified stiffness of supporting beam may be used to account for most of the T beam effect. The facts concerning such questions as this can probably be obtained satisfactorily only by laboratory and field tests.

Certain problems involve the so-called infinitely long slab. For practical purposes a solution for such a slab may be interpreted as applying to a long rectangular slab in which the two short edges are so remote from the portion considered that the boundary conditions at these edges have a negligible effect. Similarly, a solution for the so-called semi-infinite slab may be interpreted as applying to an area near one end of a long rectangular slab. The boundary conditions on three edges affect such a solution, but the fourth edge is assumed to be sufficiently remote to have a negligible effect.

In the analysis of the infinitely long slab, simply supported on two edges, material use has been made of Nádai's form of the potential function* $\phi$, proportional to the moment sum, through which the bending and twisting moments, shears and reactions become expressible in finite form. It will be shown in Chapters III and IV that this function and its derivatives may be similarly utilized in a number of problems involving long slabs supporting concentrated loads.

In Chapters V and VI rectangular slabs, simply supported on two opposite edges and having either two or three longitudinal stringers, are treated. The solutions obtained in these chapters are combined in various ways in Chapter VII to give additional solutions or to give more convenient forms of the previous ones.

2. *Acknowledgment.*—The investigation of reinforced concrete bridge slabs is being conducted in the Engineering Experiment Station with the coöperation of the United States Bureau of Public Roads and the Illinois Division of Highways. The project is under the administrative direction of Dean M. L. Enger, Director of the Engineering Experiment Station, Professor W. C. Huntington, Head of the Department of Civil Engineering and Professor

*A. Nádai, Die elastischen Platten, 1925, p. 87.

F. B. SEELY, Head of the Department of Theoretical and Applied Mechanics.

The investigation is under the general supervision of F. E. RICHART, Research Professor of Engineering Materials, with the work divided into two general classifications: (1) experimental, having R. W. KLUGE, Special Research Associate in Theoretical and Applied Mechanics, in immediate charge, and (2) analytical, having N. M. NEWMARK, Research Assistant Professor of Civil Engineering, in immediate charge.

The program of the investigation is guided by an Advisory Committee whose personnel is as follows: representing the United States Bureau of Public Roads, E. F. KELLEY, Chief, Division of Tests, and A. L. GEMENY, Senior Structural Engineer; representing the Illinois Division of Highways, ERNST LIEBERMAN, Chief Highway Engineer, and A. BENESCH, Engineer of Grade Separations; representing the University of Illinois, PROFESSORS RICHART and NEWMARK.

Acting in the capacity of Consultants to the Advisory Committee are W. M. WILSON, Research Professor of Structural Engineering, and T. C. SHEDD, Professor of Structural Engineering, both of the University of Illinois. Prior to July 1937, HARDY CROSS, Chairman of the Department of Civil Engineering, Yale University, formerly Professor of Structural Engineering, University of Illinois, served as a Consultant to the Advisory Committee.

The author appreciates the assistance he has obtained from discussions with members of the staff and from the guidance of the committee. He is especially indebted to Professor Newmark for the numerous helpful criticisms and suggestions which he has contributed to this study. Detailed computations have been made on various sections of the analytical work by Special Research Graduate Assistants H. A. LEPPER, JR., E. D. OLSON, W. A. RENNER, and CARL ROHDE, and by the late O. F. SLONIKER who was at the time an Assistant in Physics.

3. *Notation.*—The following notation has been adopted for the purpose of this bulletin:

$w$ = deflection of slab, positive downward. In special cases subscripts or primes may be used with $w$ to designate particular deflection functions. In these cases the symbols used to represent the corresponding bending moments bear a related mark of identification

$x, y$ = horizontal rectangular coördinates as defined in figures

$z$ = deflection of beam, positive downward

$a, b, u, v, s$ = dimensions defined for each solution in the corresponding figure

$h$ = thickness of slab

$E$ = modulus of elasticity of the material of the slab

$E_1, E_2, \ldots$ = moduli of elasticity of the materials of the supporting beams

$\mu$ = Poisson's ratio of the material of the slab

$I_1, I_2, \ldots$ = moments of inertia of cross sectional areas of supporting beams

$N = \dfrac{Eh^3}{12(1 - \mu^2)}$, measure of stiffness of the slab

$H_1, H_2, \ldots = \dfrac{E_1 I_1}{aN}, \dfrac{E_2 I_2}{aN}, \ldots$, dimensionless quantities defining relative stiffnesses of beams to slab

$P$ = concentrated load

$Q$ = concentrated reaction on supporting beam

$p$ = distributed load per unit of area, positive when acting downward on the slab

$q$ = line load per unit of length, positive when acting downward on a supporting beam

$V_x, V_y$ = vertical shear per unit of length, acting on sections normal to the $x$ and $y$ axes respectively, positive on a rectangular element of a slab when acting upward on the side of the element having the smaller value of $x$ or $y$ respectively

$M_x, M_y$ = bending moments per unit of length, acting on sections normal to the $x$ and $y$ axes respectively, positive when producing compression at the top of the slab. In particular cases the moments will be indicated as $M_x^{(0)}$, $M_y^{(0)}$, or $M_x^{(1)}$, $M_y^{(1)}$, meaning that these moments have been obtained from particular deflection functions $w_0$ or $w_1$

$M_{xy}$ = twisting moment per unit of length, acting on sections normal to the $x$ and $y$ axes respectively, positive when producing compression at the top of the slab in the direction of the line $x = y$

$M_{\text{beam}}$ = bending moment in a beam, positive when it produces compression in the top

$R_x, R_y$ = reactions per unit of length of slab, acting on sections normal to the $x$ and $y$ axes respectively, positive in the same sense as the corresponding shears

$R_c$ = concentrated reaction at corner, as shown in Fig. 1

$\sigma_x, \sigma_y$ = normal components of stress, in directions of $x$ and $y$ respectively

$\tau_{xy}$ = shearing component of stress, in plane normal to $x$ and in direction of $y$, or in plane normal to $y$ and in direction of $x$

$n$ = an integer. A particular summation is to be carried out for the integral values indicated under the summation sign

$$\alpha = \frac{n\pi}{a}$$

$$\beta = \alpha b = \frac{n\pi b}{a}$$

$\epsilon$ = an infinitesimally small positive quantity or distance

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, Laplace's operator in two variables

Various other quantities are defined in the text where they are used or in Appendix B.

## II. Review of Fundamental Equations

4. *The Ordinary Theory of Slabs.*—The equations pertaining to the ordinary theory of flexure of slabs will not be derived here since their derivation is available in a number of places* in the technical literature. It should be stated, however, that the ordinary theory is based upon the assumption that every line drawn through the slab normal to its middle surface before the slab is loaded remains straight and normal to the deflected middle surface after the slab is loaded. The theoretical results obtained under this assumption are applicable to bridge slabs except in the vicinity of a concentrated load.

Nádai† found the maximum stress under a loaded circular area by a special theory applicable to a thick circular slab and Westergaard‡ converted Nádai's results into expressions like those from the ordinary theory by the use of an "equivalent diameter" of a uniformly loaded circular area. These results are valid when the load is sufficiently remote from a support, as it must be to produce maximum

*See, for example: A. Nádai, Die elastischen Platten, 1925, p. 20; H. M. Westergaard and W. A. Slater, Moments and Stresses in Slabs, Proc. Am. Conc. Inst., V. 17, 1921, p. 415; H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 2.

†A. Nádai, Die elastischen Platten, 1925, p. 308.

‡H. M. Westergaard, Stresses in Concrete Pavements Computed by Theoretical Analysis, Public Roads, V. 7, No. 2, April, 1926, p. 25.

moment, but the results do not apply when the load is very near a support, the position for maximum shear.

The usual assumptions are made that the slab is homogeneous, isotropic, and elastic, and, further, that the slab is of constant thickness and of the proportions ordinarily encountered in bridge slabs. Supporting beams are assumed to exert only vertical forces upon the slab.

The fundamental equations are next stated. If $w$ designates the deflection and $p$ designates the load, the differential equation governing the deflection of the neutral surface of the slab is

$$N\nabla^4 w = N\nabla^2\nabla^2 w = p \tag{1}$$

in which Laplace's operator for two variables is denoted by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ , \tag{2}$$

and $N$, a measure of the stiffness of the slab, is

$$N = \frac{Eh^3}{12\,(1 - \mu^2)}\ . \tag{3}$$

The bending and twisting moments are given by the equations:

$$\left.\begin{aligned} M_x &= -N\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -N\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -N\,(1 - \mu)\,\frac{\partial^2 w}{\partial x \partial y}\ . \end{aligned}\right\} \tag{4}$$

At the bottom of the slab the tensile stresses, $\sigma_x$ in the direction of $x$ and $\sigma_y$ in the direction of $y$, and the shearing stress, $\tau_{xy}$ in the directions of $x$ and $y$, are given by the simple relations

$$\sigma_x = \frac{6M_x}{h^2} \quad , \quad \sigma_y = \frac{6M_y}{h^2} \quad , \quad \tau_{xy} = \frac{6M_{xy}}{h^2} \ , \tag{5}$$

where $h$ is the thickness of the slab.

The shears are

$$V_x = -N \frac{\partial}{\partial x} (\nabla^2 w) \quad , \quad V_y = -N \frac{\partial}{\partial y} (\nabla^2 w) \ . \tag{6}$$

The reactions may be stated in terms of the shears and the twisting moment or in terms of the deflection by means of the equations:

$$\left.\begin{aligned} R_x &= V_x + \frac{\partial M_{xy}}{\partial y} = -N \left[ \frac{\partial^3 w}{\partial x^3} + (2 - \mu) \frac{\partial^3 w}{\partial x \partial y^2} \right] \\ R_y &= V_y + \frac{\partial M_{xy}}{\partial x} = -N \left[ \frac{\partial^3 w}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x^2 \partial y} \right]. \end{aligned}\right\} \tag{7}$$

In addition there exists at every corner, having edges normal to the axes of $x$ and $y$, a concentrated reaction $R_c$ equal to twice the value of $M_{xy}$ at the particular corner. The positive sense of each of the corner reactions,

$$R_c = 2M_{xy} \ , \tag{8}$$

is as shown in Fig. 1.

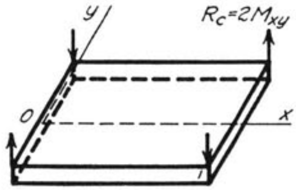



FIG. 1

A solution of the differential Equation (1), satisfying the particular boundary conditions of a given type of slab, therefore yields, through its derivatives, all of the ordinary relations of moments, shears and reactions.

5. *Solution for the Infinitely Long Slab Supporting a Concentrated Load.*—The slab and loading are shown in Fig. 2 where the edges

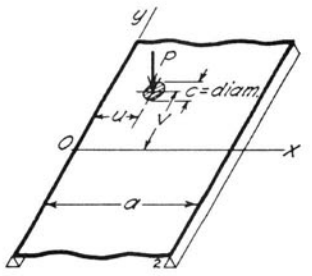



FIG. 2

$x = 0$ and $x = a$ are assumed to be simply supported and where the slab is assumed to extend sufficiently far in the directions of positive and negative $y$ that the deflection of the slab practically vanishes before additional supports or edges are reached. The deflection of the slab is then given by the equation*

$$w_0 = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (1 + \alpha|y - v|)\, e^{-\alpha|y-v|} \sin \alpha u \sin \alpha x \qquad (9)$$

where

$$\alpha = \frac{n\pi}{a}.$$

Nádai† defined a function $\phi_0$ by means of the equation

$$\phi_0 = N\nabla^2 w_0 = -\frac{P}{\pi} \sum_{1,2,3,\cdots} \frac{1}{n}\, e^{-\alpha|y-v|} \sin \alpha u \sin \alpha x \qquad (10)$$

and derived‡ an expression for $\phi_0$ in finite form, namely,

$$\phi_0 = \frac{P}{4\pi} \log_e \frac{B_0}{A_0} \qquad (11)$$

*See, for example, A. Nádai, Die elastischen Platten, 1925, p. 85; or H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 6. Equation (9) has been adjusted for the origin of coördinates shown in Fig. 2 and the absolute value of $(y - v)$ has been introduced to extend the region of applicability of the equation to the entire slab. See Appendix A for notes on the differentiation with respect to $y$ of functions involving $|y - v|$.

†A. Nádai, Die elastischen Platten, 1925, p. 87. (See previous footnote.)

‡A. Nádai, Die elastischen Platten, 1925, p. 89.

where

$$\left.\begin{matrix} A_0 \\ B_0 \end{matrix}\right\} = \cosh \frac{\pi (y - v)}{a} - \cos \frac{\pi (x \pm u)}{a}. \tag{12}$$

By differentiation of (10) and (11) with respect to $x$ and $y$ one obtains

$$\left.\begin{aligned} \frac{\partial \phi_0}{\partial x} &= -\frac{P}{a} \sum_{1,2,3,\cdots} e^{-\alpha|y-v|} \sin \alpha u \cos \alpha x \\ &= \frac{P}{4a} \left[ \frac{1}{B_0} \sin \frac{\pi (x - u)}{a} - \frac{1}{A_0} \sin \frac{\pi (x + u)}{a} \right] \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} \frac{\partial \phi_0}{\partial y} &= \frac{P \operatorname{sgn} (y - v)}{a} \sum_{1,2,3,\cdots} e^{-\alpha|y-v|} \sin \alpha u \sin \alpha x \\ &= \frac{P}{4a} \left( \frac{1}{B_0} - \frac{1}{A_0} \right) \sinh \frac{\pi (y - v)}{a}. \end{aligned}\right\} \tag{14}^*$$

Nádai† showed that the curvatures of the slab may be expressed through the function $\phi_0$ and its derivatives by means of the equations

$$\left.\begin{aligned} 2N \frac{\partial^2 w_0}{\partial x^2} &= \phi_0 - (y - v) \frac{\partial \phi_0}{\partial y} \\ 2N \frac{\partial^2 w_0}{\partial y^2} &= \phi_0 + (y - v) \frac{\partial \phi_0}{\partial y} \\ 2N \frac{\partial^2 w_0}{\partial x \partial y} &= (y - v) \frac{\partial \phi_0}{\partial x}. \end{aligned}\right\} \tag{15}$$

*See Appendix A for a discussion of the change of sign represented by the function sgn $(y - v)$, namely,

$$\operatorname{sgn} (y - v) = \text{algebraic sign of } (y - v) = \begin{cases} +1 \text{ for } y > v \\ -1 \text{ for } y < v \end{cases}.$$

†A. Nádai, Die elastischen Platten, 1925, p. 86.

The equations for moment then become

$$\left.\begin{aligned} M_x^{(0)} &= -\frac{1+\mu}{2}\phi_0 + \frac{1-\mu}{2}(y-v)\frac{\partial\phi_0}{\partial y} \\ M_y^{(0)} &= -\frac{1+\mu}{2}\phi_0 - \frac{1-\mu}{2}(y-v)\frac{\partial\phi_0}{\partial y} \\ M_{xy}^{(0)} &= -\frac{1-\mu}{2}(y-v)\frac{\partial\phi_0}{\partial x}. \end{aligned}\right\} \tag{16}$$

Substitution of $\phi_0$ and its derivatives into (16) gives the following equations for the moments at any point $x$, $y$:

$$\left.\begin{aligned} &\left.\begin{matrix} M_x^{(0)} \\ M_y^{(0)} \end{matrix}\right\} = -\frac{(1+\mu)P}{8\pi}\log_e\frac{B_0}{A_0} \pm \frac{(1-\mu)P(y-v)}{8a}\left(\frac{1}{B_0}-\frac{1}{A_0}\right)\sinh\frac{\pi(y-v)}{a} \\ &M_{xy}^{(0)} = -\frac{(1-\mu)P(y-v)}{8a}\left[\frac{1}{\mathrm{B}_0}\sin\frac{\pi(x-u)}{a} - \frac{1}{A_0}\sin\frac{\pi(x+u)}{a}\right]. \end{aligned}\right\} \tag{17}$$

Near the load the moments given by these equations are not valid. The special formulas given by Westergaard* may be used to obtain bending moments which lead to the proper maximum stresses under the load. With the origin of coördinates and the position of the load given by Fig. 2, the special formulas for the resultant moments under the load become

$$\left.\begin{aligned} M_x^{(0)}\Big]_{\substack{x=u\\y=v}} &= \frac{(1+\mu)P}{4\pi}\log_e\left(\frac{4a}{\pi c_1}\sin\frac{\pi u}{a}\right) + \frac{P}{4\pi} \\ M_y^{(0)}\Big]_{\substack{x=u\\y=v}} &= M_x^{(0)}\Big]_{\substack{x=u\\y=v}} - \frac{(1-\mu)P}{4\pi} \\ M_{xy}^{(0)}\Big]_{\substack{x=u\\y=v}} &= 0, \end{aligned}\right\} \tag{18}$$

*H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 8. See Westergaard's Equations 57 to 62 inclusive.

Holl gives special treatment to the problem of finding the correct stresses in the vicinity of a concentrated load on a rectangular slab with pinned-free edges. Use is made of the analysis of a thick square slab with simply supported edges. See D. L. Holl, Analysis of Thin Rectangular Plates Supported on Opposite Edges, Bulletin 129, Iowa Engineering Experiment Station, Iowa State College, 1936, p. 34.

in which

$$c_1 = \begin{cases} 2\left(\sqrt{0.4c^2 + h^2} - 0.675h\right) & \text{for} \quad c < 3.45h \\ c & \text{for} \quad c > 3.45h \end{cases}$$

where

$c$ = diameter of a small circular area assumed to be uniformly loaded,

$h$ = thickness of slab.

In much of the work that follows, the results are presented in the form of corrections to be added to the corresponding basic quantities given in the foregoing.

## III. The Infinitely Long Slab With Simply Supported Edges

6. *Concentrated Load at Any Point on the Infinitely Long Slab Having a Rigid Cross Beam.*—Figure 3 shows the slab and the orienta-

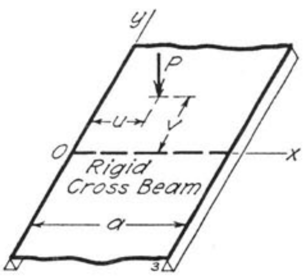



Fig. 3

tion of the axes relative to the cross beam and the load. Without the cross beam, the deflection of the slab is given by $w_0$ of Equation (9). The deflection due to a line load of unknown distribution upon the line $y = 0$ is given by the equation

$$w_1 = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{a_n}{n^3} (1 + \alpha|y|)\, e^{-\alpha|y|} \sin \alpha u \sin \alpha x \qquad (19)$$

where the load distribution factor $a_n$ is to be determined by the boundary condition at $y = 0$. The total deflection is then

$$w = w_0 + w_1.$$

Since the beam is rigid, the total deflection must vanish on the line $y = 0$. Therefore

$$w\Big]_{y=0} = \Big[w_0 + w_1\Big]_{y=0} = 0$$

for every value of $x$. Substitution of (9) and (19) into this equation gives

$$a_n = -(1 + \alpha v)\, e^{-\alpha v}. \tag{20}$$

Therefore the part of the deflection which is due to the beam reaction is

$$\left.\begin{aligned} w_1 &= -\frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3}(1+\alpha v)(1+\alpha|y|)e^{-\alpha(v+|y|)} \sin \alpha u \sin \alpha x \\ &= -\frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3}[1+\alpha(v+|y|)]e^{-\alpha(v+|y|)} \sin \alpha u \sin \alpha x \\ &\quad -\frac{Pav}{2\pi^2 N} \sum_{1,2,3,\cdots} \frac{1}{n^2}\alpha|y|e^{-\alpha(v+|y|)} \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{21}$$

As in the previous solution, one may define a function

$$\left.\begin{aligned} \phi_1 = N\nabla^2 w_1 &= \frac{P}{\pi} \sum_{1,2,3,\cdots} \frac{1}{n}\, e^{-\alpha(v+|y|)} \sin \alpha u \sin \alpha x \\ &\quad + \frac{Pv}{a} \sum_{1,2,3,\cdots} e^{-\alpha(v+|y|)} \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{22}$$

A set of equations,

$$\left.\begin{aligned} 2N\frac{\partial^2 w_1}{\partial x^2} &= \phi_1 - y\frac{\partial \phi_1}{\partial y}, \\ 2N\frac{\partial^2 w_1}{\partial y^2} &= \phi_1 + y\frac{\partial \phi_1}{\partial y}, \\ 2N\frac{\partial^2 w_1}{\partial x \partial y} &= y\frac{\partial \phi_1}{\partial x}, \end{aligned}\right\} \tag{23}$$

corresponding to (15), may be verified by differentiation of (21) and (22) and substitution into (23).

From (22) one may write, by analogy with (10), (11) and (14), the finite form of $\phi_1$, namely,

$$\phi_1 = \frac{P}{4\pi} \log_e \frac{A_1}{B_1} + \frac{Pv}{4a}\left(\frac{1}{B_1} - \frac{1}{A_1}\right) \sinh \frac{\pi (v + |y|)}{a} \tag{24}$$

where

$$\left.\begin{matrix} A_1 \\ B_1 \end{matrix}\right\} = \cosh \frac{\pi (v + |y|)}{a} - \cos \frac{\pi (x \pm u)}{a}. \tag{25}$$

To verify this solution, note the boundary condition

$$\left.\frac{\partial^2 w}{\partial x^2}\right]_{y=0} = \left[\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_1}{\partial x^2}\right]_{y=0} = 0$$

which, from (15) and (23), becomes

$$\left.\phi_1\right]_{y=0} = -\left[\phi_0 + v\frac{\partial \phi_0}{\partial y}\right]_{y=0}.$$

Substitution of $\phi_0$ into this equation gives

$$\left.\phi_1\right]_{y=0} = \frac{P}{4\pi} \log_e \frac{A}{B} + \frac{Pv}{4a}\left(\frac{1}{B} - \frac{1}{A}\right) \sinh \frac{\pi v}{a} \tag{26}$$

where

$$\left.\begin{matrix} A \\ B \end{matrix}\right\} = \cosh \frac{\pi v}{a} - \cos \frac{\pi (x \pm u)}{a}. \tag{27}$$

It is easily verified, then, that Equation (24) for $\phi_1$ has the following properties:

(1) It satisfies $\nabla^2 \phi_1 = 0$ at all points.
(2) It vanishes at $x = 0$, $x = a$ and $y = \pm \infty$.
(3) It satisfies (26) at $y = 0$.

Equation (24), satisfying both the differential equation and the boundary conditions, is therefore the correct solution.

The derivatives of $\phi_1$ needed in Equations (23) are

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial x} &= \frac{P}{4a}\left[\frac{\sin\frac{\pi(x+u)}{a}}{A_1^2}\left(A_1+\frac{\pi v}{a}\sinh\frac{\pi(v+|y|)}{a}\right)\right. \\ &\qquad \left.-\frac{\sin\frac{\pi(x-u)}{a}}{B_1^2}\left(B_1+\frac{\pi v}{a}\sinh\frac{\pi(v+|y|)}{a}\right)\right], \\ \frac{\partial \phi_1}{\partial y} &= \frac{P\operatorname{sgn} y}{4a}\left(\frac{1}{A_1}-\frac{1}{B_1}\right)\left[\sinh\frac{\pi(v+|y|)}{a}-\frac{\pi v}{a}\cosh\frac{\pi(v+|y|)}{a}\right. \\ &\qquad \left.+\frac{\pi v}{a}\left(\frac{1}{A_1}+\frac{1}{B_1}\right)\sinh^2\frac{\pi(v+|y|)}{a}\right]. \end{aligned}\right\} \quad (28)^*$$

The corrective moments due to $w_1$ are

$$\left.\begin{aligned} M_x^{(1)} &= -\frac{1+\mu}{2}\phi_1+\frac{1-\mu}{2}y\frac{\partial\phi_1}{\partial y} \\ M_y^{(1)} &= -\frac{1+\mu}{2}\phi_1-\frac{1-\mu}{2}y\frac{\partial\phi_1}{\partial y} \\ M_{xy}^{(1)} &= -\frac{1-\mu}{2}y\frac{\partial\phi_1}{\partial x}, \end{aligned}\right\} \quad (29)$$

in which $\phi_1$ and its derivatives are given by (24) and (28).

*See Appendix A for a discussion of the change of sign represented by the function "sgn $y$," namely,
$\operatorname{sgn} y = \text{algebraic sign of } y = \begin{cases} +1 \text{ for } y > 0 \\ -1 \text{ for } y < 0. \end{cases}$

Equations (29) are of particular interest under the load, where they become

$$\left.\begin{aligned}
M_x^{(1)}\Big]_{\substack{x=u\\y=v}} &= -\frac{(1+\mu)P}{8\pi}\log_e\frac{A'}{B'} - \frac{Pv}{4a}\left(\frac{1}{B'}-\frac{1}{A'}\right)\Bigg[\sinh\frac{2\pi v}{a} \\
&\qquad + \frac{\pi v(1-\mu)}{2a}\left(1+\frac{\sinh^2\frac{2\pi v}{a}}{A'}\right)\Bigg], \\
M_y^{(1)}\Big]_{\substack{x=u\\y=v}} &= -\frac{(1+\mu)P}{8\pi}\log_e\frac{A'}{B'} - \frac{Pv}{4a}\left(\frac{1}{B'}-\frac{1}{A'}\right)\Bigg[\mu\sinh\frac{2\pi v}{a} \\
&\qquad - \frac{\pi v(1-\mu)}{2a}\left(1+\frac{\sinh^2\frac{2\pi v}{a}}{A'}\right)\Bigg], \\
M_{xy}^{(1)}\Big]_{\substack{x=u\\y=v}} &= -\frac{(1-\mu)Pv}{8a}\,\frac{\sin\frac{2\pi u}{a}}{A'}\left(1+\frac{\pi v}{a}\,\frac{\sinh\frac{2\pi v}{a}}{A'}\right),
\end{aligned}\right\} \quad (30)$$

where

$$A' = \cosh\frac{2\pi v}{a} - \cos\frac{2\pi u}{a}, \qquad B' = \cosh\frac{2\pi v}{a} - 1. \quad (31)$$

Equations (30) give corrective moments under the load which are to be added to the corresponding moments in the infinitely long slab without a cross beam, as given by Equations (18), in order to obtain the resultant moments under the load. Numerical values of corrective bending moments computed from (30) are given and discussed in Section 8.

The resultant bending moments over the beam are

$$M_x\Big]_{y=0} = \Big[M_x^{(0)} + M_x^{(1)}\Big]_{y=0}, \qquad M_y\Big]_{y=0} = \Big[M_y^{(0)} + M_y^{(1)}\Big]_{y=0}.$$

One finds

$$\left.\begin{aligned} M_y\Big]_{y=0} &= -\frac{Pv}{4a}\left(\frac{1}{B} - \frac{1}{A}\right)\sinh\frac{\pi v}{a}, \\ M_x\Big]_{y=0} &= \mu M_y\Big]_{y=0}, \end{aligned}\right\} \tag{32}$$

where $A$ and $B$ are given by (27). There is also a twisting moment in the slab over the beam which is given by the formula

$$\left.\begin{aligned} M_{xy}\Big]_{y=0} &= \Big[M_{xy}^{(0)} + M_{xy}^{(1)}\Big]_{y=0} = M_{xy}^{(0)}\Big]_{y=0} \\ &= \frac{(1-\mu)Pv}{8a}\left(\frac{\sin\dfrac{\pi(x+u)}{a}}{A} - \frac{\sin\dfrac{\pi(x-u)}{a}}{B}\right). \end{aligned}\right\} \tag{33}$$

The maximum bending moment in the slab over the beam is, therefore, not normal to the beam at every point along its length.

The upward reaction of the cross beam upon the slab, or the downward load on the beam, is

$$q = 2V_y^{(1)}\Big]_{y=\epsilon} = -2N\left[\frac{\partial}{\partial y}(\nabla^2 w_1)\right]_{y=\epsilon} = -2\frac{\partial \phi_1}{\partial y}\Big]_{y=\epsilon}$$

where $\epsilon$ is an infinitesimally small positive distance. Therefore

$$q = \frac{P}{2a}\left(\frac{1}{B} - \frac{1}{A}\right)\left[\sinh\frac{\pi v}{a} - \frac{\pi v}{a}\cosh\frac{\pi v}{a} + \frac{\pi v}{2a}\left(\frac{1}{B} + \frac{1}{A}\right)\left(\cosh\frac{2\pi v}{a} - 1\right)\right]$$

where $A$ and $B$ are given by (27).

From a subsequent solution, wherein the beam is permitted a certain flexibility (see Equation 46), it may be shown that the equation for the bending moment in the beam, when the beam becomes infinitely rigid, is

$$\left.\begin{aligned} M_{\text{beam}} &= \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} (1 + \alpha v)\, e^{-\alpha v} \sin \alpha u \sin \alpha x \\ &= \frac{Pv}{2\pi} \log_e \frac{\cosh \frac{\pi v}{a} - \cos \frac{\pi (x+u)}{a}}{\cosh \frac{\pi v}{a} - \cos \frac{\pi (x-u)}{a}} \\ &\quad + \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} e^{-\alpha v} \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{34}$$

7. *Concentrated Load at the Center of the Infinitely Long Slab Having a Rigid Cross Beam.*—When $u = a/2$ the load is in the center of the

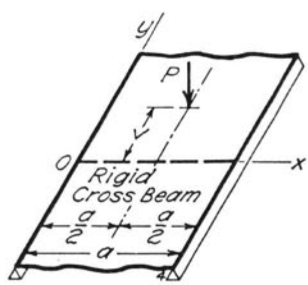



FIG. 4

slab as shown in Fig. 4. Equations (30), which express the corrective moments under the load, then become

$$\left.\begin{aligned}
M_x^{(1)}\Big]_{\substack{x=a/2\\y=v}} &= -\frac{(1+\mu)P}{4\pi}\log_e\coth\frac{\pi v}{a}\\
&\quad -\frac{Pv}{2a}\left[1+\frac{(1-\mu)\pi v}{2a}\coth\frac{2\pi v}{a}\right]\operatorname{csch}\frac{2\pi v}{a},\\
M_y^{(1)}\Big]_{\substack{x=a/2\\y=v}} &= -\frac{(1+\mu)P}{4\pi}\log_e\coth\frac{\pi v}{a}\\
&\quad -\frac{Pv}{2a}\left[\mu-\frac{(1-\mu)\pi v}{2a}\coth\frac{2\pi v}{a}\right]\operatorname{csch}\frac{2\pi v}{a},\\
M_{xy}^{(1)}\Big]_{\substack{x=a/2\\y=v}} &= 0.
\end{aligned}\right\}\quad(35)$$

The resultant bending moment $M_y$ in the slab over the beam becomes, when $u = a/2$ in (32),

$$M_y\Big]_{y=0} = -\frac{Pv}{a}\,\frac{\sinh\frac{\pi v}{a}\sin\frac{\pi x}{a}}{\cosh\frac{2\pi v}{a}+\cos\frac{2\pi x}{a}}. \qquad (36)$$

Since the corrective twisting moment in the slab over the beam is zero, according to (35), the resultant twisting moment is given completely by the effect of a concentrated load on the infinitely long slab without the cross beam. The equation for the twisting moment in the slab over the beam is, therefore,

$$M_{xy}\Big]_{y=0} = M_{xy}^{(0)}\Big]_{y=0} = \frac{(1-\mu)Pv}{2a}\,\frac{\cosh\frac{\pi v}{a}\cos\frac{\pi x}{a}}{\cosh\frac{2\pi v}{a}+\cos\frac{2\pi x}{a}}. \qquad (37)$$

Westergaard* gives numerical values and curves obtained from an

*H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 12-15. See Westergaard's Equation 75, Figures 12 and 14, and Table 5.

expression like (37) except for a difference of sign and changed coördinates.

Where the twisting moment in the slab over the beam is not zero, its effect is to produce a maximum bending moment, $M_{\max}$, having a direction which is inclined to the axis of the beam. The maximum moment is negative and, since $M_x = \mu M_y$ over the beam, one obtains in the usual way,

$$M_{\max} = \frac{1 + \mu}{2} M_y - \sqrt{\left(\frac{1 - \mu}{2}\right)^2 M_y^2 + M_{xy}^2}. \tag{38}$$

If $\theta$ is the angle between the $y$ axis and the direction of $M_{\max}$, positive when counter-clockwise, then

$$\tan 2\theta = \frac{2M_{xy}}{(1 - \mu) M_y} = -\coth \frac{\pi v}{a} \cot \frac{\pi x}{a}. \tag{39}$$

The upward reaction of the beam on the slab is given by the formula

$$\left.\begin{aligned} q = \frac{2P}{a} \frac{\sin \frac{\pi x}{a}}{\cosh \frac{2\pi v}{a} + \cos \frac{2\pi x}{a}} \Bigg[ & \sinh \frac{\pi v}{a} \\ & - \left(\frac{\pi v}{a} \cosh \frac{\pi v}{a}\right) \left(1 - 2 \frac{\cosh \frac{2\pi v}{a} - 1}{\cosh \frac{2\pi v}{a} + \cos \frac{2\pi x}{a}}\right) \Bigg]. \end{aligned}\right\} \tag{40}$$

Equation (34), which expresses the bending moment in the beam, becomes, when $u = a/2$,

$$\left.\begin{aligned} M_{\text{beam}} &= \frac{2Pa}{\pi^2} \sum_{1,3,5,\cdots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} (1 + \alpha v) e^{-\alpha v} \sin \alpha x \\ &= \frac{Pv}{2\pi} \log_e \frac{\cosh \frac{\pi v}{a} + \sin \frac{\pi x}{a}}{\cosh \frac{\pi v}{a} - \sin \frac{\pi x}{a}} + \frac{2Pa}{\pi^2} \sum_{1,3,5,\cdots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} e^{-\alpha v} \sin \alpha x. \end{aligned}\right\} \tag{41}$$

At the center of the beam this moment becomes

$$\left.\begin{aligned} \max M_{\text{beam}} &= \frac{2Pa}{\pi^2} \sum_{1,3,5,\cdots} \frac{1}{n^2}(1+\alpha v)\,e^{-\alpha v} \\ &= \frac{Pv}{\pi} \log_e \coth \frac{\pi v}{2a} + \frac{2Pa}{\pi^2} \sum_{1,3,5,\cdots} \frac{1}{n^2}\,e^{-\alpha v}. \end{aligned}\right\} \quad (42)$$

8. *Numerical Computations of Moments in the Infinitely Long Slab Having a Rigid Cross Beam and Supporting a Concentrated Load.*— Numerical values* of the corrective bending moments $M_x^{(1)}$ and $M_y^{(1)}$ under the load, computed from (30) and (35) for $\mu = 0.15$, are

TABLE 1

CORRECTIVE MOMENTS UNDER THE LOAD DUE TO A RIGID CROSS BEAM

Concentrated load on the infinitely long slab of span $a$ having a rigid cross beam. The load is at a distance $u$ from one of the edges of the slab and at a distance $v$ from the cross beam. Numerical values of $M_x^{(1)}$ and $M_y^{(1)}$ in the table are corrective moments under the load, due to the presence of the beam, and were computed from Equations (30) and (35). The corrective moments are to be added to the corresponding moments found under the load on the infinitely long slab without the cross beam. Poisson's ratio, $\mu = 0.15$. Curves of moments are shown in Fig. 5 and Fig. 6.

| $\frac{u}{a}$ | $\frac{v}{a}$ | $\frac{M_x^{(1)}}{P}$ | $\frac{M_y^{(1)}}{P}$ | $\frac{u}{a}$ | $\frac{v}{a}$ | $\frac{M_x^{(1)}}{P}$ | $\frac{M_y^{(1)}}{P}$ |
|---|---|---|---|---|---|---|---|
| 0.1 or 0.9 | 0.05 | −0.1547 | −0.0627 | 0.4 or 0.6 | 0.05 | −0.2610 | −0.1602 |
| | 0.10 | −0.0869 | −0.0193 | | 0.10 | −0.1966 | −0.0978 |
| | 0.15 | −0.0522 | −0.0066 | | 0.15 | −0.1580 | −0.0629 |
| | 0.20 | −0.0334 | −0.0024 | | 0.20 | −0.1299 | −0.0400 |
| | 0.30 | −0.0159 | +0.0001 | | 0.30 | −0.0892 | −0.0135 |
| | 0.40 | −0.0087 | +0.0006 | | 0.40 | −0.0607 | −0.0014 |
| | 0.50 | −0.0051 | +0.0007 | | 0.50 | −0.0406 | +0.0033 |
| | 0.60 | −0.0031 | +0.0006 | | 0.60 | −0.0267 | +0.0045 |
| | 0.80 | −0.0012 | +0.0004 | | 0.80 | −0.0109 | +0.0033 |
| | 1.00 | −0.0005 | +0.0002 | | 1.00 | −0.0043 | +0.0017 |
| 0.2 or 0.8 | 0.05 | −0.2162 | −0.1170 | 0.5 | 0.05 | −0.2656 | −0.1647 |
| | 0.10 | −0.1501 | −0.0582 | | 0.10 | −0.2014 | −0.1022 |
| | 0.15 | −0.1101 | −0.0298 | | 0.15 | −0.1630 | −0.0669 |
| | 0.20 | −0.0821 | −0.0149 | | 0.20 | −0.1350 | −0.0434 |
| | 0.30 | −0.0473 | −0.0026 | | 0.30 | −0.0942 | −0.0154 |
| | 0.40 | −0.0283 | +0.0011 | | 0.40 | −0.0651 | −0.0021 |
| | 0.50 | −0.0174 | +0.0020 | | 0.50 | −0.0441 | +0.0033 |
| | 0.60 | −0.0109 | +0.0020 | | 0.60 | −0.0292 | +0.0048 |
| | 0.80 | −0.0043 | +0.0013 | | 0.80 | −0.0121 | +0.0036 |
| | 1.00 | −0.0016 | +0.0007 | | 1.00 | −0.0047 | +0.0019 |
| 0.3 or 0.7 | 0.05 | −0.2460 | −0.1456 | | | | |
| | 0.10 | −0.1812 | −0.0840 | | | | |
| | 0.15 | −0.1420 | −0.0506 | | | | |
| | 0.20 | −0.1135 | −0.0299 | | | | |
| | 0.30 | −0.0736 | −0.0084 | | | | |
| | 0.40 | −0.0478 | +0.0001 | | | | |
| | 0.50 | −0.0309 | +0.0030 | | | | |
| | 0.60 | −0.0199 | +0.0035 | | | | |
| | 0.80 | −0.0080 | +0.0024 | | | | |
| | 1.00 | −0.0031 | +0.0012 | | | | |

*Numerical values of the hyperbolic functions were obtained from the British Association for the Advancement of Science, Mathematical Tables, V. 1, London, 1931.

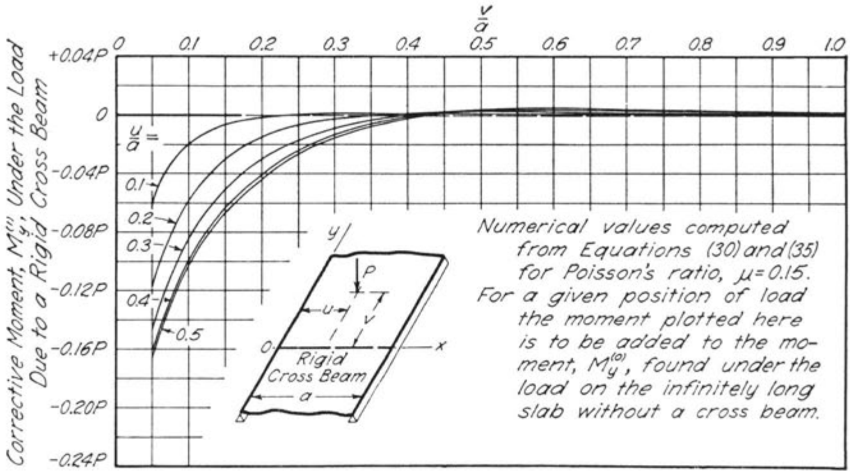



FIG. 5

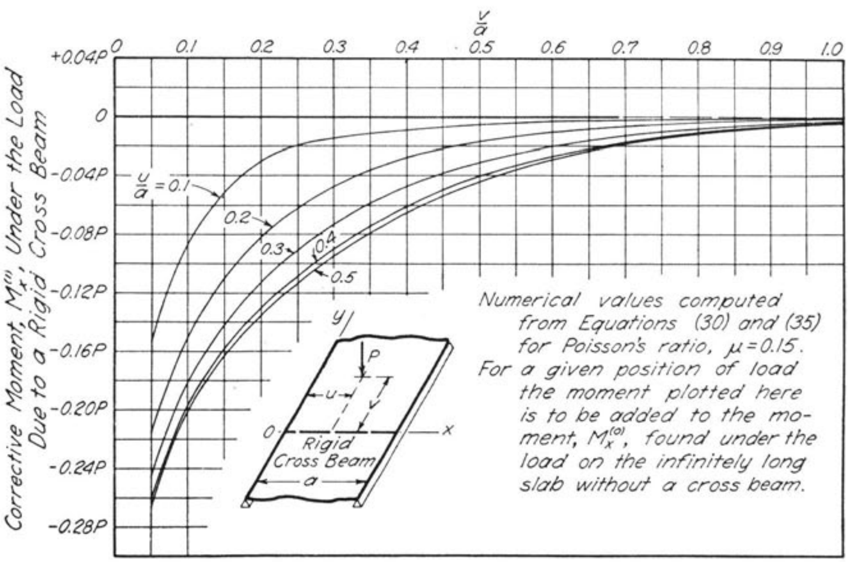



FIG. 6

given in Table 1 for values of $u/a$ varying by tenths across the span, and for values of $v/a$ varying from 0.05 to 1.0. The corresponding curves are shown in Figs. 5 and 6. It is apparent that the correction to $M_y$ diminishes very rapidly as the distance from the load to the cross beam increases, and is practically zero for values of $v/a$ greater than 0.4. The moment $M_x$ is affected over a wider range of values of $v/a$; however, for every position of the load across the span, the reduction in $M_x$ at $v/a = 0.5$ is less than 0.05 $P$.

As an example of the influence of the beam in reducing the bending moments under the load, consider the specific case of a central load $P$, uniformly distributed over a small circle of diameter $c = a/10$, on a slab having a ratio of span to thickness, $a/h$, equal to 12 and assuming that Poisson's ratio $\mu = 0.15$. Without the cross beam the moments per unit of length in the slab under the load are*

$$M_x^{(0)}\Big]_{\substack{x=u\\y=v}} = 0.3154P$$

$$M_y^{(0)}\Big]_{\substack{x=u\\y=v}} = (0.3154 - 0.0676)\ P = 0.2478\ P.$$

Adding to these moments the corrections given in Table 1 and Figs. 5 and 6 for $u/a = 0.5$ gives the final moments shown in Fig. 7.

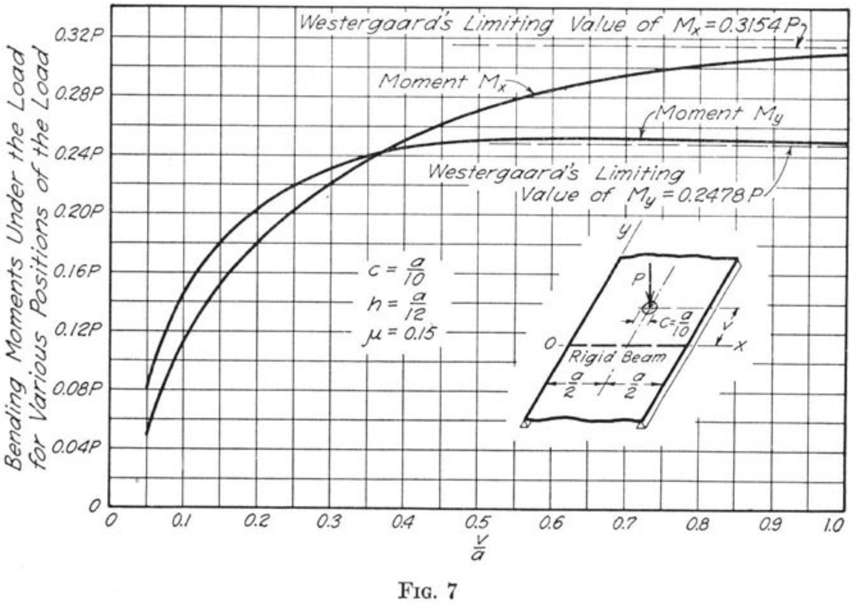



FIG. 7

*H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 9, Table 1.

TABLE 2

BENDING MOMENT $M_y$ IN THE SLAB OVER THE CROSS BEAM

The slab is infinitely long and has a rigid cross beam as shown in Fig. 3. Numerical values were computed from Equation (32). The corresponding curves are shown in Fig. 8. These moments are independent of Poisson's ratio.

| $\frac{v}{a}$ | $\frac{x}{a}$ | $-\frac{M_y}{P}$ when $u/a =$ 0.1 | $\frac{v}{a}$ | $\frac{x}{a}$ | $-\frac{M_y}{P}$ when $u/a =$ 0.2 | $\frac{v}{a}$ | $\frac{x}{a}$ | $-\frac{M_y}{P}$ when $u/a =$ 0.3 | $\frac{v}{a}$ | $\frac{x}{a}$ | $-\frac{M_y}{P}$ when $u/a =$ 0.4 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 0.05 | 0.0637 | 0.05 | 0.10 | 0.0275 | 0.05 | 0.10 | 0.0069 | 0.05 | 0.10 | 0.0027 |
| | 0.10 | 0.1498 | | 0.15 | 0.0764 | | 0.20 | 0.0302 | | 0.20 | 0.0082 |
| | 0.15 | 0.0734 | | 0.20 | 0.1567 | | 0.25 | 0.0782 | | 0.30 | 0.0309 |
| | 0.20 | 0.0275 | | 0.25 | 0.0776 | | 0.30 | 0.1580 | | 0.35 | 0.0788 |
| | 0.30 | 0.0069 | | 0.30 | 0.0302 | | 0.35 | 0.0786 | | 0.40 | 0.1584 |
| | 0.40 | 0.0027 | | 0.40 | 0.0082 | | 0.40 | 0.0309 | | 0.45 | 0.0789 |
| | 0.50 | 0.0013 | | 0.50 | 0.0034 | | 0.50 | 0.0086 | | 0.50 | 0.0312 |
| | 0.60 | 0.0007 | | 0.60 | 0.0017 | | 0.60 | 0.0036 | | 0.60 | 0.0087 |
| | 0.70 | 0.0004 | | 0.70 | 0.0009 | | 0.70 | 0.0018 | | 0.70 | 0.0036 |
| | 0.80 | 0.0002 | | 0.80 | 0.0005 | | 0.80 | 0.0009 | | 0.80 | 0.0017 |
| | 0.90 | 0.0001 | | 0.90 | 0.0002 | | 0.90 | 0.0004 | | 0.90 | 0.0007 |
| 0.10 | 0.05 | 0.0783 | 0.10 | 0.10 | 0.0636 | 0.10 | 0.10 | 0.0224 | 0.10 | 0.10 | 0.0097 |
| | 0.10 | 0.1273 | | 0.15 | 0.1152 | | 0.20 | 0.0733 | | 0.20 | 0.0273 |
| | 0.15 | 0.1053 | | 0.20 | 0.1497 | | 0.25 | 0.1220 | | 0.30 | 0.0760 |
| | 0.20 | 0.0636 | | 0.25 | 0.1197 | | 0.30 | 0.1546 | | 0.35 | 0.1241 |
| | 0.30 | 0.0224 | | 0.30 | 0.0733 | | 0.35 | 0.1233 | | 0.40 | 0.1562 |
| | 0.40 | 0.0097 | | 0.40 | 0.0273 | | 0.40 | 0.0760 | | 0.45 | 0.1245 |
| | 0.50 | 0.0049 | | 0.50 | 0.0124 | | 0.50 | 0.0289 | | 0.50 | 0.0769 |
| | 0.60 | 0.0027 | | 0.60 | 0.0065 | | 0.60 | 0.0133 | | 0.60 | 0.0293 |
| | 0.70 | 0.0016 | | 0.70 | 0.0036 | | 0.70 | 0.0069 | | 0.70 | 0.0133 |
| | 0.80 | 0.0009 | | 0.80 | 0.0020 | | 0.80 | 0.0036 | | 0.80 | 0.0065 |
| | 0.90 | 0.0004 | | 0.90 | 0.0009 | | 0.90 | 0.0016 | | 0.90 | 0.0027 |
| 0.20 | 0.05 | 0.0479 | 0.20 | 0.10 | 0.0781 | 0.20 | 0.10 | 0.0474 | 0.20 | 0.10 | 0.0266 |
| | 0.10 | 0.0795 | | 0.15 | 0.1103 | | 0.20 | 0.1047 | | 0.20 | 0.0627 |
| | 0.15 | 0.0875 | | 0.20 | 0.1269 | | 0.25 | 0.1304 | | 0.30 | 0.1138 |
| | 0.20 | 0.0781 | | 0.25 | 0.1230 | | 0.30 | 0.1422 | | 0.35 | 0.1375 |
| | 0.30 | 0.0474 | | 0.30 | 0.1047 | | 0.35 | 0.1348 | | 0.40 | 0.1477 |
| | 0.40 | 0.0266 | | 0.40 | 0.0627 | | 0.40 | 0.1138 | | 0.45 | 0.1390 |
| | 0.50 | 0.0153 | | 0.50 | 0.0357 | | 0.50 | 0.0682 | | 0.50 | 0.1170 |
| | 0.60 | 0.0091 | | 0.60 | 0.0208 | | 0.60 | 0.0388 | | 0.60 | 0.0697 |
| | 0.70 | 0.0055 | | 0.70 | 0.0123 | | 0.70 | 0.0222 | | 0.70 | 0.0388 |
| | 0.80 | 0.0032 | | 0.80 | 0.0069 | | 0.80 | 0.0123 | | 0.80 | 0.0208 |
| | 0.90 | 0.0014 | | 0.90 | 0.0032 | | 0.90 | 0.0055 | | 0.90 | 0.0091 |
| 0.40 | 0.10 | 0.0315 | 0.40 | 0.10 | 0.0472 | 0.40 | 0.10 | 0.0466 | 0.40 | 0.10 | 0.0381 |
| | 0.15 | 0.0417 | | 0.20 | 0.0780 | | 0.20 | 0.0853 | | 0.20 | 0.0750 |
| | 0.20 | 0.0472 | | 0.25 | 0.0845 | | 0.30 | 0.1065 | | 0.30 | 0.1054 |
| | 0.25 | 0.0484 | | 0.30 | 0.0853 | | 0.35 | 0.1085 | | 0.35 | 0.1152 |
| | 0.30 | 0.0466 | | 0.35 | 0.0816 | | 0.40 | 0.1054 | | 0.40 | 0.1200 |
| | 0.40 | 0.0381 | | 0.40 | 0.0750 | | 0.45 | 0.0982 | | 0.45 | 0.1193 |
| | 0.50 | 0.0284 | | 0.50 | 0.0582 | | 0.50 | 0.0885 | | 0.50 | 0.1137 |
| | 0.60 | 0.0201 | | 0.60 | 0.0419 | | 0.60 | 0.0665 | | 0.60 | 0.0924 |
| | 0.70 | 0.0135 | | 0.70 | 0.0284 | | 0.70 | 0.0458 | | 0.70 | 0.0665 |
| | 0.80 | 0.0083 | | 0.80 | 0.0174 | | 0.80 | 0.0284 | | 0.80 | 0.0419 |
| | 0.90 | 0.0039 | | 0.90 | 0.0083 | | 0.90 | 0.0135 | | 0.90 | 0.0201 |

As the distance $u$ from the beam to the load increases, the moments are seen to rise in magnitude until they approach the values for the infinitely long slab without a cross beam.

Numerical values of the bending moments $M_y$ in the slab over the beam, computed from (32), are given in Table 2, and are shown in Fig. 8 for various positions of the load on lines at distances of 0.1, 0.2, 0.3 and 0.4 of the span from one edge. Similar moments, computed from (36), are given in Table 3 and are shown in Fig. 9 for the

TABLE 3

BENDING MOMENT $M_y$ IN THE SLAB OVER THE CROSS BEAM

Concentrated load in the center of the infinitely long slab having a rigid cross beam. Numerical values were computed from Equation (36). Curves have been plotted in Fig. 9 where the coördinate axes are also shown. These moments are independent of Poisson's ratio.

| $\frac{v}{a}$ | $\frac{x}{a}$ | $-\frac{M_y}{P}$ | $\frac{v}{a}$ | $\frac{x}{a}$ | $-\frac{M_y}{P}$ |
|---|---|---|---|---|---|
| 0.05 | 0.10 | 0.0013 | 0.40 | 0.10 | 0.0284 |
| | 0.20 | 0.0034 | | 0.20 | 0.0582 |
| | 0.30 | 0.0086 | | 0.30 | 0.0885 |
| | 0.40 | 0.0312 | | 0.40 | 0.1137 |
| | 0.45 | 0.0789 | | 0.45 | 0.1212 |
| | 0.50 | 0.1585 | | 0.50 | 0.1239 |
| 0.10 | 0.10 | 0.0049 | 0.50 | 0.10 | 0.0287 |
| | 0.20 | 0.0124 | | 0.20 | 0.0568 |
| | 0.30 | 0.0289 | | 0.30 | 0.0825 |
| | 0.40 | 0.0769 | | 0.40 | 0.1015 |
| | 0.45 | 0.1247 | | 0.45 | 0.1068 |
| | 0.50 | 0.1566 | | 0.50 | 0.1086 |
| 0.15 | 0.10 | 0.0099 | 0.60 | 0.10 | 0.0265 |
| | 0.20 | 0.0241 | | 0.20 | 0.0516 |
| | 0.30 | 0.0508 | | 0.30 | 0.0730 |
| | 0.40 | 0.1043 | | 0.40 | 0.0879 |
| | 0.45 | 0.1375 | | 0.45 | 0.0919 |
| | 0.50 | 0.1534 | | 0.50 | 0.0933 |
| 0.20 | 0.10 | 0.0153 | 0.80 | 0.10 | 0.0197 |
| | 0.20 | 0.0357 | | 0.20 | 0.0377 |
| | 0.30 | 0.0682 | | 0.30 | 0.0523 |
| | 0.40 | 0.1170 | | 0.40 | 0.0619 |
| | 0.45 | 0.1397 | | 0.45 | 0.0644 |
| | 0.50 | 0.1491 | | 0.50 | 0.0652 |
| 0.30 | 0.10 | 0.0241 | 1.00 | 0.10 | 0.0133 |
| | 0.20 | 0.0522 | | 0.20 | 0.0253 |
| | 0.30 | 0.0863 | | 0.30 | 0.0349 |
| | 0.40 | 0.1213 | | 0.40 | 0.0411 |
| | 0.45 | 0.1334 | | 0.45 | 0.0428 |
| | 0.50 | 0.1378 | | 0.50 | 0.0433 |

load on the center of the span and for values of $v/a$ varying from 0.05 to 1.0.

It is significant that the limiting value of $M_y$ in the slab over the beam is $-P/(2\pi)$ regardless of the distance $u$ between the load and the edge of the slab. This limiting value is approached as the load tends to cross the beam.

When the load is near the cross beam it is permissible to interpret the slab as a bridge floor which is continuous over stringers and floor beams. In this case the cross beam corresponds to a floor beam, and the distance $a$ corresponds to the span of the slab between stringers. The validity of such an interpretation arises from the condition that the continuity of the slab across the floor beam is the principal factor

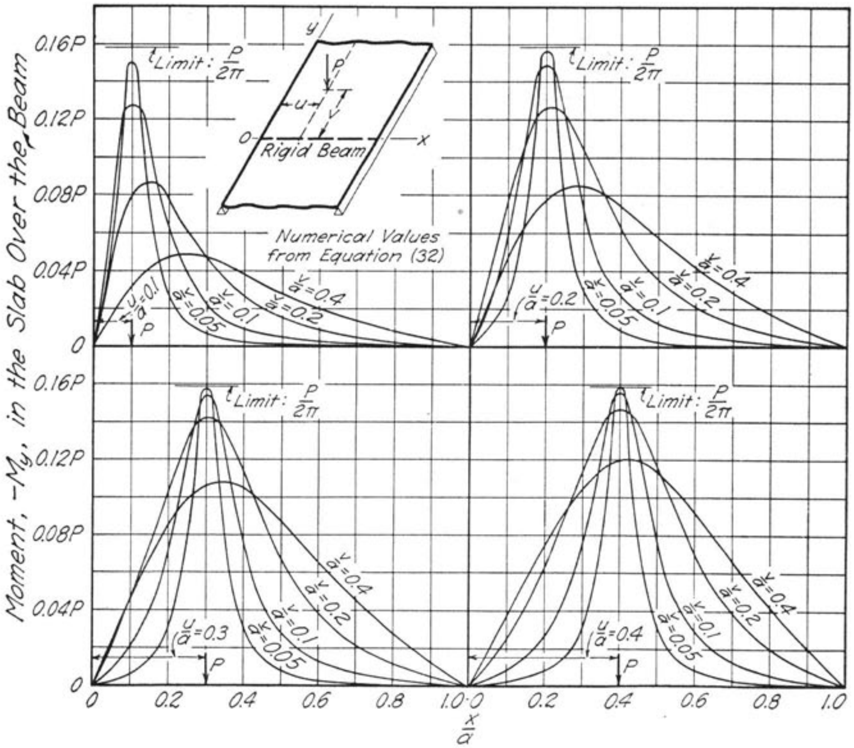



FIG. 8

entering such a solution; the importance of the continuity of the slab across the stringers diminishes as the load approaches the floor beam. A limiting value of $-P/(2\pi)$ per unit length for $M_y$ in the slab over the floor beam is therefore reached as a single load crosses the beam. The effects of other loads are not included in this value.

To illustrate the effect of the twisting moment in the slab over the beam, numerical values of $M_{\max}$ were computed for $\mu = 0.15$ from (38) for a central position of the load on the span, and for ratios of $v/a$ equal to 0.1 and 0.2. The results have been plotted in Fig. 10, where the directions of $M_{\max}$ are also shown at a number of points across the span. It is apparent that the curves of maximum moment do not rise above the greatest value of $M_y$ for the positions of load considered.

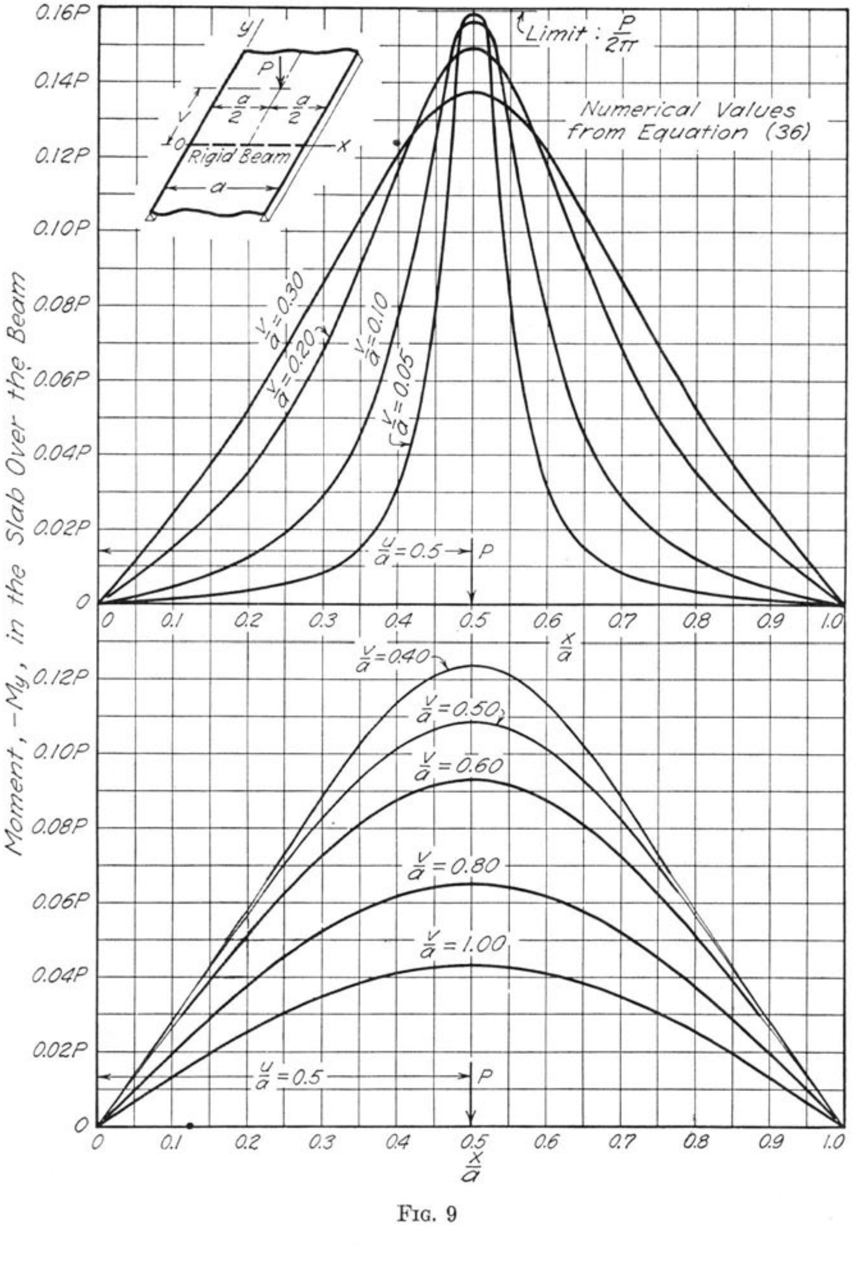
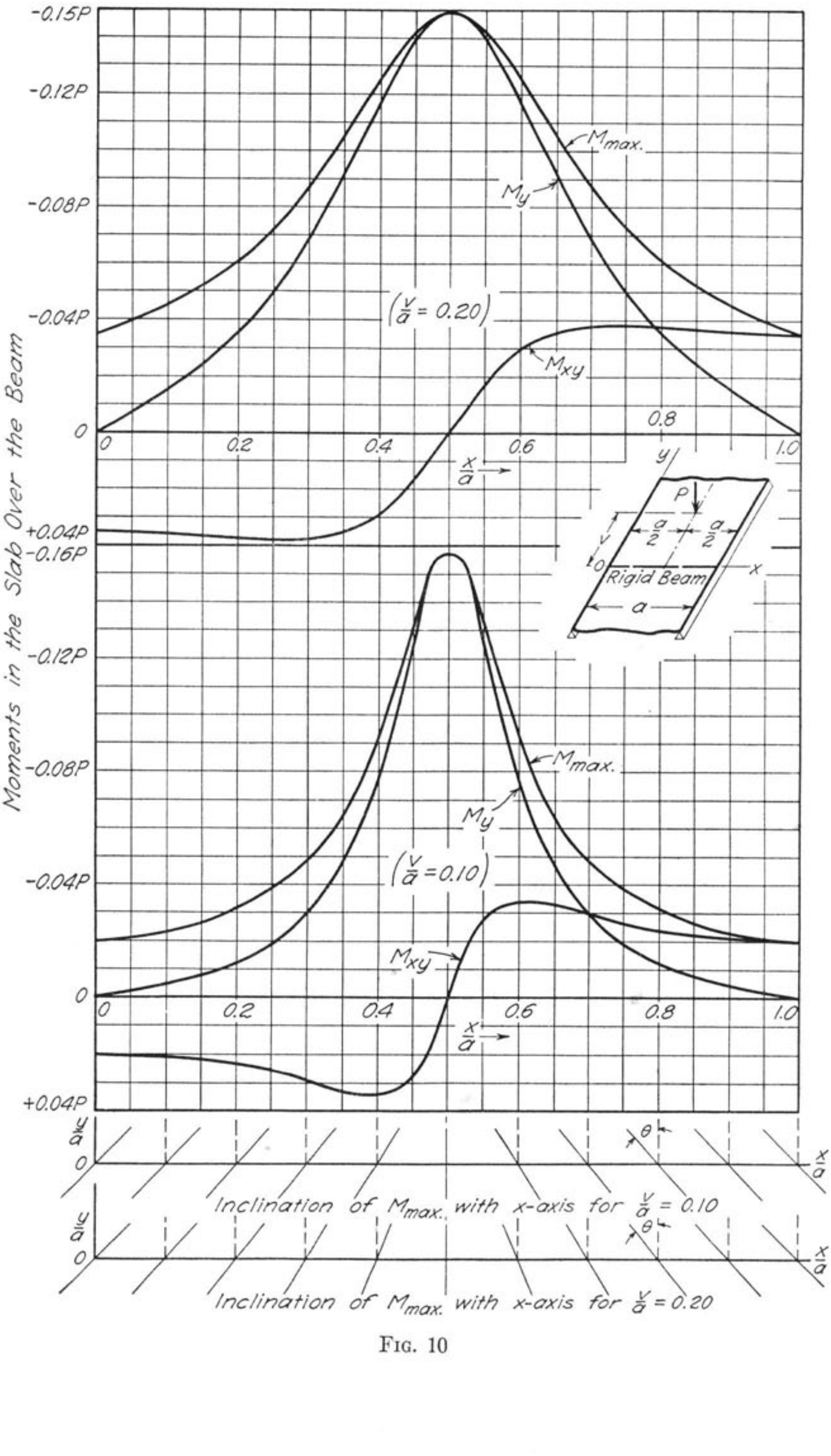

Limit : $\frac{P}{2\pi}$
Numerical Values from Equation (36)
Rigid Beam
Moment, $-M_y$, in the Slab Over the Beam
$\frac{v}{a}=0.30$
$\frac{v}{a}=0.20$
$\frac{v}{a}=0.10$
$\frac{v}{a}=0.05$
$\frac{u}{a}=0.5$
$\frac{v}{a}=0.40$
$\frac{v}{a}=0.50$
$\frac{v}{a}=0.60$
$\frac{v}{a}=0.80$
$\frac{v}{a}=1.00$
$\frac{x}{a}$

FIG. 9



-0.15P
-0.12P
-0.08P
-0.04P
0
+0.04P
-0.16P
-0.12P
-0.08P
-0.04P
0
+0.04P
Moments in the Slab Over the Beam
$M_{max.}$
$M_y$
$M_{xy}$
$\left(\frac{v}{a} = 0.20\right)$
$\left(\frac{v}{a} = 0.10\right)$
0 0.2 0.4 0.6 0.8 1.0
$\frac{x}{a}$
P
$\frac{a}{2}$ $\frac{a}{2}$
Rigid Beam
a
v
y
x
$\frac{y}{a}$
$\theta$
Inclination of $M_{max.}$ with x-axis for $\frac{v}{a} = 0.10$
Inclination of $M_{max.}$ with x-axis for $\frac{v}{a} = 0.20$


FIG. 10

9. *Concentrated Load at Any Point on the Infinitely Long Slab Having a Flexible Cross Beam.**—The slab and loading are shown in Fig. 11. The deflection of the slab without the cross beam is

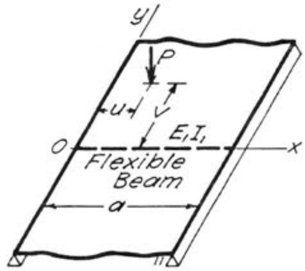



FIG. 11

given by $w_0$ of Equation (9). The deflection due to a line load of unknown distribution on the line $y = 0$ is given by (19) where $a_n$ is to be determined from the boundary condition:

$$q = E_1 I_1 \frac{d^4 z}{dx^4} = \left[ V_y^{(1)} \right]_{y=\epsilon} - \left[ V_y^{(1)} \right]_{y=-\epsilon} = -2N \left[ \frac{\partial}{\partial y} (\nabla^2 w_1) \right]_{y=\epsilon}.$$

In this equation $q$ is the downward load on the beam, $E_1$ and $I_1$ are the modulus of elasticity and moment of inertia of the beam, respectively, $z$ is the deflection of the beam given by the equation

$$z = w \Big]_{y=0} = \Big[ w_0 + w_1 \Big]_{y=0},$$

$V_y^{(1)}$ is the shear due to $w_1$, and $\epsilon$ is an infinitesimally small positive distance.

From the application of the boundary condition to (19), one finds

$$a_n = -\frac{(1 + \alpha v)\, e^{-\alpha v}}{1 + \dfrac{4}{n\pi H_1}} \tag{43}$$

where

$$H_1 = \frac{E_1 I_1}{aN}.$$

*Bay obtained solutions for a similar slab supporting either a uniformly distributed load over the entire slab or a uniform line load directly over the beam. See Hermann Bay, "Die Kreuzweise gespannten Fahrbahnplatten bei stählernen Brücken," Beton und Eisen, 33, Heft 14, July, 1934, p. 222-224.

The total deflection of the slab is

$$w = w_0 + w_1$$

where $w_0$ is the deflection of the infinitely long slab, without the cross beam, loaded as shown in Fig. 11, and $w_1$ is a corrective deflection given by the equation

$$w_1 = -\frac{Pa^2}{2\pi^3 N}\sum_{1,2,3,\cdots}\frac{1}{n^3}\frac{(1+\alpha v)}{1+\dfrac{4}{n\pi H_1}}(1+\alpha|y|)e^{-\alpha(v+|y|)}\sin\alpha u\sin\alpha x. \quad (44)$$

The deflection of the beam is

$$z = \frac{Pa^2}{2\pi^3 N}\sum_{1,2,3,\cdots}\frac{1}{n^3}\frac{(1+\alpha v)\,e^{-\alpha v}}{1+\dfrac{n\pi H_1}{4}}\sin\alpha u\sin\alpha x. \quad (45)$$

The curvatures due to $w_1$ are

$$\frac{\partial^2 w_1}{\partial x^2} = \frac{P}{2\pi N}\sum_{1,2,3,\cdots}\frac{1}{n}\frac{(1+\alpha v)}{1+\dfrac{4}{n\pi H_1}}(1+\alpha|y|)e^{-\alpha(v+|y|)}\sin\alpha u\sin\alpha x,$$

$$\frac{\partial^2 w_1}{\partial y^2} = \frac{P}{2\pi N}\sum_{1,2,3,\cdots}\frac{1}{n}\frac{(1+\alpha v)}{1+\dfrac{4}{n\pi H_1}}(1-\alpha|y|)e^{-\alpha(v+|y|)}\sin\alpha u\sin\alpha x.$$

Under the load, at $x = u$, $y = v$, these become

$$\left.\frac{\partial^2 w_1}{\partial x^2}\right]_{\substack{x=u\\y=v}} = \frac{P}{2\pi N}\sum_{1,2,3,\cdots}\frac{1}{n}\frac{(1+\alpha v)^2\,e^{-2\alpha v}}{1+\dfrac{4}{n\pi H_1}}\sin^2\alpha u,$$

$$\left.\frac{\partial^2 w_1}{\partial y^2}\right]_{\substack{x=u\\y=v}} = \frac{P}{2\pi N}\sum_{1,2,3,\cdots}\frac{1}{n}\frac{(1-\alpha^2 v^2)\,e^{-2\alpha v}}{1+\dfrac{4}{n\pi H_1}}\sin^2\alpha u,$$

which lead to the corrective moments under the load.

The bending moment in the beam is

$$M_{\text{beam}} = -E_1 I_1 \frac{d^2 z}{dx^2} = \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} \frac{(1 + \alpha v)\, e^{-\alpha v}}{1 + \dfrac{4}{n\pi H_1}} \sin \alpha u \sin \alpha x \quad (46)$$

which becomes a maximum when $u = x = a/2$ and $v = 0$. Under these conditions

$$\text{max. } M_{\text{beam}} = \frac{2Pa}{\pi^2} \sum_{1,3,5,\cdots} \frac{1}{n^2} \frac{1}{1 + \dfrac{4}{n\pi H_1}}$$

$$= \frac{Pa}{4} - \frac{2Pa}{\pi^2} \sum_{1,3,5,\cdots} \frac{1}{n^2} \frac{1}{1 + \dfrac{n\pi H_1}{4}}.$$

In the slab over the beam the curvatures due to $w_1$ are

$$\left.\frac{\partial^2 w_1}{\partial x^2}\right]_{y=0} = \left.\frac{\partial^2 w_1}{\partial y^2}\right]_{y=0} = \frac{P}{2\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} \frac{(1 + \alpha v)\, e^{-\alpha v}}{1 + \dfrac{4}{n\pi H_1}} \sin \alpha u \sin \alpha x,$$

giving corrective moments in the slab over the beam

$$\left.M_x^{(1)}\right]_{y=0} = \left.M_y^{(1)}\right]_{y=0} = -\frac{(1+\mu)P}{2\pi} \sum_{1,2,3\cdots} \frac{1}{n} \frac{(1+\alpha v)e^{-\alpha v}}{1 + \dfrac{4}{n\pi H_1}} \sin \alpha u \sin \alpha x.$$

When the load is over the beam, $v = 0$, and the resultant moments in the slab over the beam become

$$\left.M_x\right]_{\substack{y=0\\v=0}} = \left.M_y\right]_{\substack{y=0\\v=0}} = \left.\frac{(1 + \mu)}{aH_1} M_{\text{beam}}\right]_{v=0}$$

$$= \frac{(1 + \mu)\, P}{2\pi} \sum_{1,2,3,\cdots} \frac{1}{n} \frac{1}{1 + \dfrac{n\pi H_1}{4}} \sin \alpha u \sin \alpha x.$$

10. *The Infinitely Long Slab Supporting a Concentrated Load Midway Between Two Flexible Cross Beams.*—Because of symmetry about the $x$ axis as shown in Fig. 12, it is sufficient to obtain the deflection

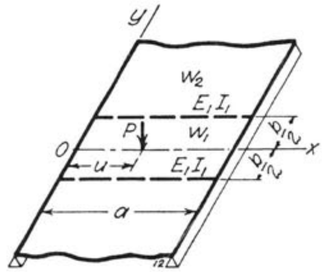



FIG. 12

functions for $y > 0$. Letting $w_1$ and $w_2$ represent the total deflection of the slab for $b/2 > y > 0$ and for $y > b/2$ respectively, one may write

$$w_0 = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (1 + \alpha y)\, e^{-\alpha y} \sin \alpha u \sin \alpha x, \tag{47}$$

$$w_1 = w_0 - \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{k}{n^3} \left[ \left(1 + \frac{\alpha b}{2}\right) \cosh \alpha y - \alpha y \sinh \alpha y \right] e^{-\frac{\alpha b}{2}} \sin \alpha u \sin \alpha x \tag{48}$$

and

$$w_2 = w_0 - \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{k}{n^3} \left[ (1 + \alpha y) \cosh \frac{\alpha b}{2} - \frac{\alpha b}{2} \sinh \frac{\alpha b}{2} \right] e^{-\alpha y} \sin \alpha u \sin \alpha x \tag{49}$$

where $k$ is to be determined from the boundary condition. In Equations (48) and (49) the correction to $w_0$ has been expressed as the sum of the deflections of the infinitely long slab,

$$w' = -\frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{k}{n^3} (1 + \alpha|y - b/2|) e^{-\alpha|y - b/2|} \sin \alpha u \sin \alpha x \tag{50}$$

and

$$w'' = -\frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{k}{n^3} (1 + \alpha|y + b/2|) e^{-\alpha|y + b/2|} \sin \alpha u \sin \alpha x, \tag{51}$$

due to loads distributed along the lines $y = b/2$ and $y = -b/2$, respectively.

The boundary condition, from which $k$ is to be determined, is

$$q = E_1 I_1 \frac{d^4 z}{dx^4} = \left[ V'_y \right]_{y=b/2+\epsilon} - \left[ V'_y \right]_{y=b/2-\epsilon}$$

$$= -2N \left[ \frac{\partial}{\partial y} (\nabla^2 w') \right]_{y=b/2+\epsilon}$$

where $q$ is the load on the beam, positive downward, $E_1$ and $I_1$ are the modulus of elasticity and moment of inertia of the beam, respectively, $z$ is the downward deflection of either beam, and $\epsilon$ is an infinitesimally small positive distance. From this boundary condition one finds

$$k = \frac{1 + \dfrac{\alpha b}{2}}{(1 + \alpha b) e^{-\frac{\alpha b}{2}} + \left(1 + \dfrac{4}{n\pi H_1}\right) e^{\frac{\alpha b}{2}}}. \tag{52}$$

The deflection of each beam is

$$z = w_1 \Big]_{y=b/2} = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \frac{4}{n\pi H_1} k \sin \alpha u \sin \alpha x, \tag{53}$$

and the bending moment in each beam is

$$M_{\text{beam}} = -E_1 I_1 \frac{d^2 z}{dx^2} = \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} k \sin \alpha u \sin \alpha x. \tag{54}$$

If $M_x^{(0)}$ and $M_y^{(0)}$ designate the bending moments due to the load $P$ on the infinitely long slab without cross beams, the resultant bending moments under the load are

$$\left.\begin{aligned} M_x \Big]_{\substack{x=u \\ y=0}} &= M_x^{(0)} \Big]_{\substack{x=u \\ y=0}} - \frac{P}{\pi} \sum_{1,2,3,\cdots} \frac{k e^{-\frac{\alpha b}{2}}}{n} \left[ (1+\mu) + (1-\mu) \frac{\alpha b}{2} \right] \sin^2 \alpha u, \\ M_y \Big]_{\substack{x=u \\ y=0}} &= M_y^{(0)} \Big]_{\substack{x=u \\ y=0}} - \frac{P}{\pi} \sum_{1,2,3,\cdots} \frac{k e^{-\frac{\alpha b}{2}}}{n} \left[ (1+\mu) - (1-\mu) \frac{\alpha b}{2} \right] \sin^2 \alpha u, \end{aligned}\right\} \tag{55}$$

the terms given by the series being the effects of the cross beams.

The solution may be modified for rigid beams by letting $H_1 = \infty$. This special case is treated more fully in Section 39 for a central position of the load on the panel.

11. *The Infinitely Long Slab Having Three Uniformly Spaced Flexible Cross Beams and Supporting a Concentrated Load Over the Center Beam.*—The slab is shown in Fig. 13. Using the same method

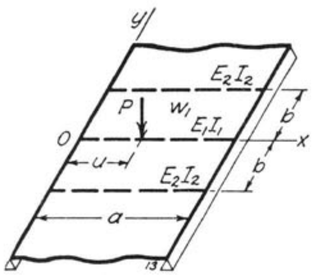



FIG. 13

as was used in the previous examples with flexible beams, one finds the deflection of the slab for $b > y > 0$ to be

$$\left.\begin{aligned} w_1 = \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{a_n}{n^4} (b_n \cosh \alpha y &- \sinh \alpha y \\ &+ \alpha y \cosh \alpha y - c_n\, \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x \end{aligned}\right\} \quad (56)$$

where

$$\left.\begin{aligned} a_n &= \frac{1}{\dfrac{4}{n\pi H_1} + b_n}, \\ b_n &= \frac{n\pi H_2\, [1 - (1 + 2\alpha b + 2\alpha^2 b^2)\, e^{-2\alpha b}] + 4}{n\pi H_2\, [1 + (1 + 2\alpha b)\, e^{-2\alpha b}] + 4}, \\ c_n &= \frac{n\pi H_2\, (1 - e^{-2\alpha b}) + 4}{n\pi H_2\, [1 + (1 + 2\alpha b)\, e^{-2\alpha b}] + 4}. \end{aligned}\right\} \quad (57)$$

The deflection of the center beam is

$$\left.\begin{aligned} z_1 = w_1\Big]_{y=0} &= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4}\, a_n b_n \sin \alpha u \sin \alpha x \\ &= z_0 - \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4} \frac{4}{n\pi H_1}\, a_n \sin \alpha u \sin \alpha x \end{aligned}\right\} \tag{58}$$

in which $z_0$ is the deflection of the center beam when it supports the concentrated load but is detached from the slab; that is

$$\left.\begin{aligned} z_0 &= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4} \sin \alpha u \sin \alpha x \\ &= \frac{P\,(a-u)\,x}{6E_1 I_1 a}\,(2au - x^2 - u^2) \qquad \text{for} \qquad x < u \\ &= \frac{P\,(a-x)\,u}{6E_1 I_1 a}\,(2ax - u^2 - x^2) \qquad \text{for} \qquad x > u. \end{aligned}\right\} \tag{59}$$

The deflection of the beam at $y = b$ or at $y = -b$ is

$$z_2 = w_1\Big]_{y=b} = \frac{8Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4}\, a_n d_n \sin \alpha u \sin \alpha x \tag{60}$$

where $a_n$ is given by (57) and where

$$d_n = \frac{(1 + \alpha b)\, e^{-\alpha b}}{n\pi H_2\,[1 + (1 + 2\alpha b)\, e^{-2\alpha b}] + 4}. \tag{61}$$

The bending moment in the center beam is

$$M_{\text{beam 1}} = -E_1 I_1 \frac{d^2 z_0}{dx^2} - \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} \frac{4}{n\pi H_1}\, a_n \sin \alpha u \sin \alpha x,$$

and the bending moment in each of the other beams is

$$M_{\text{beam 2}} = \frac{8Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^4}\, a_n d_n \sin \alpha u \sin \alpha x.$$

When the load is in the center of Beam 1 the bending moment in that beam, under the load, is

$$\max M_{\text{beam }1} = \frac{Pa}{4}\left(1 - \frac{8}{\pi^2}\sum_{1,3,5,\cdots}\frac{1}{n^2}\frac{4}{n\pi H_1}a_n\right). \tag{62}$$

Formulas which express the moments in the slab are readily obtained by differentiation of (56).

## IV. The Semi-infinite Slab Carrying a Concentrated Load and Having the Long Edges Simply Supported

12. *The Finite Edge Simply Supported.*—The effect of a pair of loads on the infinitely long slab, one upward and one downward at the same distance $u$ from the $y$ axis, has been discussed by Nádai* and by Westergaard† who point out that the slab deforms as if it were cut and simply supported on a line midway between the loads. By a simple expedient, therefore, one obtains the deflection of the semi-infinite slab having simply supported edges and carrying a concentrated load.

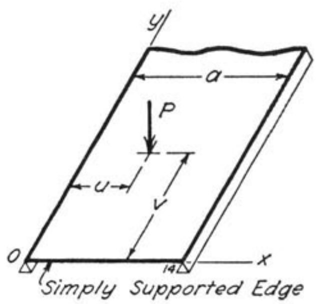



Fig. 14

The slab to be considered is shown in Fig. 14. The deflection may be composed of two parts,

$$w = w_0 - w_0', \tag{63}$$

in which $w_0$, given by Equation (9), is the deflection of an infinitely long slab due to a concentrated load at the point $x = u$, $y = v$, and

*A. Nádai, Die elastischen Platten, 1925, p. 158.

†H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 17.

$w_0'$ is the deflection of an infinitely long slab due to a concentrated load at the point $x = u$, $y = -v$. The part of the deflection $w_0'$ is given by the equation

$$w_0' = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \left[1 + \alpha (y + v)\right] e^{-\alpha(y+v)} \sin \alpha u \sin \alpha x. \quad (64)$$

A particular problem is solved, then, by taking the difference between two solutions. Westergaard's numerical values and curves, mentioned previously, may be used in each of these solutions.

13. *The Finite Edge Fixed.*—When the finite edge, coinciding with

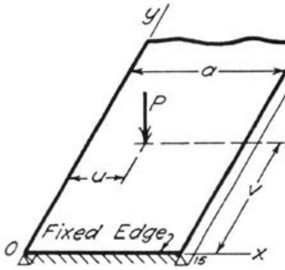



Fig. 15

the $x$ axis, is fixed as indicated in Fig. 15, one may write the expression for the deflection of the slab as the sum of three parts:

$$w = w_0 + w_0' + 2w_1 \quad (65)$$

where $w_0$, $w_0'$ and $w_1$ are given by Equations (9), (64) and (21) respectively. In (21) one may replace $|y|$ by $y$ since the slab no longer exists for $y < 0$.

Since the moments, shears and reactions are obtainable in finite form for each part of (65), they are obtainable for the entire solution. The results obtained in previous sections may be applied to this solution.

It is of interest to observe that the part of the deflection produced by the fixing of the edge, obtained as the difference between (65) and (63), namely,

$$w_f = 2\,(w_0' + w_1) = -\frac{Pav}{\pi^2 N} \sum_{1,2,3,\cdots} \frac{1}{n^2}\, \alpha y\, e^{-\alpha(v+y)} \sin \alpha u \sin \alpha x, \quad (66)$$

may be represented in finite form. Substitution of $n\pi/a$ for $\alpha$ in (66), and comparison with (10) and (11), shows that

$$w_f = -\frac{Pvy}{4\pi N} \log_e \frac{\cosh \dfrac{\pi (v + y)}{a} - \cos \dfrac{\pi (x + u)}{a}}{\cosh \dfrac{\pi (v + y)}{a} - \cos \dfrac{\pi (x - u)}{a}}. \quad (67)$$

One has the interesting result, therefore, that (67) expresses in finite form the deflection produced by the fixed-end moment and the corresponding edge reactions.

Nádai* showed in like manner that the deflection of the infinitely long slab due to a concentrated moment applied at a point on the $x$ axis, and having the same direction as $M_y$, may be represented by an expression similar to (67).

The fixed-end moment is twice as great as that given by Equations (32) and (36) for the corresponding positions of load. In other words, continuity of the slab across a rigid beam in this particular example produces a moment $M_y$ in the slab over the beam equal to 50 per cent of the fixed-end moment. Newmark† generalizes further on this representation of a continuous slab as being "50 per cent fixed" under certain conditions.

14. *The Finite Edge Free.*—When the finite edge is free, as shown

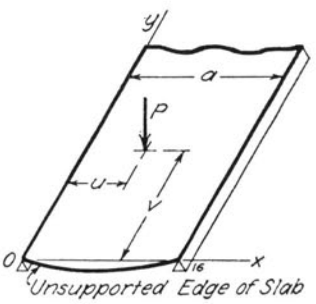



FIG. 16

in Fig. 16, the boundary conditions on the free edge are

*A. Nádai, Die elastischen Platten, 1925, p. 163.
†N. M. Newmark, "A Distribution Procedure for the Analysis of Slabs Continuous Over Flexible Beams," Bulletin 304, Engineering Experiment Station, University of Illinois, 1938, Chapter IX.

$$\left[\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right]_{y=0} = 0, \tag{68}$$

$$\left[\frac{\partial}{\partial y} (\nabla^2 w) + (1 - \mu) \frac{\partial^3 w}{\partial x^2 \partial y}\right]_{y=0} = 0, \tag{69}$$

from the requirements that the moment and reaction must vanish on the free edge.

Let the deflection again be made up of two parts,

$$w = w_0 + w_2, \tag{70}$$

the first due to the concentrated load on the infinitely long slab and the second due to the corrective moment and reaction which must be added along the line $y = 0$ to make it a free edge. Then $w_0$ is represented by (9) and $w_2$ may be represented by an equation of the form*

$$w_2 = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (a_n + b_n \alpha y)\, e^{-\alpha y} \sin \alpha u \sin \alpha x, \tag{71}$$

where $a_n$ and $b_n$ are unknown parameters to be determined by the boundary conditions.

The application of the condition equations, (68) and (69), leads to two equations in $a_n$ and $b_n$ whose simultaneous solution gives the results

$$\left.\begin{aligned} a_n &= \frac{1 - \mu}{3 + \mu} \left[\frac{4\,(1 + \mu)}{(1 - \mu)^2} + (1 + \alpha v)\right] e^{-\alpha v}, \\ b_n &= \frac{1 - \mu}{3 + \mu} (1 + 2\alpha v)\, e^{-\alpha v}. \end{aligned}\right\} \tag{72}$$

Thus

$$\left.\begin{aligned} w_2 = \frac{1 - \mu}{3 + \mu} \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \Bigg[ & \frac{4\,(1 + \mu)}{(1 - \mu)^2} \\ & + (1 + \alpha v) + (1 + 2\alpha v)\, \alpha y \Bigg] e^{-\alpha(v+y)} \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{73}$$

*A. Nádai, Die elastischen Platten, 1925, p. 69.

Equation (73) may in turn be broken up into three parts,

$$w_2 = w' + w'' + w''', \tag{74}$$

where the parts are chosen as follows:

$$w' = \frac{1-\mu}{3+\mu} \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \left[1 + \alpha\,(y+v)\right] e^{-\alpha(y+v)} \sin \alpha u \sin \alpha x \tag{75}$$

which, for $y > (-v)$, is the solution for the infinitely long slab loaded with a concentrated force $(1-\mu)P/(3+\mu)$ at the point $x = u$, $y = -v$;

$$w'' = \frac{1+\mu}{(3+\mu)\,(1-\mu)} \frac{2Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} e^{-\alpha(y+v)} \sin \alpha u \sin \alpha x, \tag{76}$$

and

$$w''' = \frac{1-\mu}{3+\mu} \frac{Pvy}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} e^{-\alpha(y+v)} \sin \alpha u \sin \alpha x. \tag{77}$$

For the first part of the correction, $w'$, all of the previous results for a concentrated load on an infinitely long slab are available. For the second part, $w''$, it may be observed that

$$\nabla^2 w'' = 0.$$

Since

$$N \frac{\partial^2 w'}{\partial x^2} = -N \frac{\partial^2 w}{\partial y^2}$$

$$= -\frac{1+\mu}{(3+\mu)\,(1-\mu)} \frac{2P}{\pi} \sum_{1,2,3,\cdots} \frac{1}{n} e^{-\alpha(y+v)} \sin \alpha u \sin \alpha x,$$

one has a relationship similar to (10) and (11), from which

$$N \frac{\partial^2 w''}{\partial x^2} = -N \frac{\partial^2 w''}{\partial y^2} = \frac{1+\mu}{(3+\mu)\,(1-\mu)} \frac{P}{2\pi} \log_e \frac{B_1}{A_1} \tag{78}$$

where $A_1$ and $B_1$ are given by (25). From (4) and (78)

$$M''_x = -M''_y = -\frac{1+\mu}{3+\mu}\frac{P}{2\pi}\log_e\frac{B_1}{A_1}. \tag{79}$$

Again, previous results may be used to determine $M''_x$ and $M''_y$, since (79) represents the moment sum, $(M_x + M_y)$, in the infinitely long slab due to a load of $2P/(3+\mu)$ at the point $x = u$, $y = -v$.

The third part of the correction, $w'''$, may be written at once in finite form, since it differs from (66) only by a constant multiplier. Thus

$$w''' = -\frac{1-\mu}{3+\mu}\frac{Pvy}{4\pi N}\log_e\frac{B_1}{A_1}. \tag{80}$$

Moments may be obtained in finite form from (80) by differentiation.

The corrective bending moments due to $w_2$, obtained as the sum of the effects of $w'$, $w''$ and $w'''$, may be summarized as follows:

$$\left.\begin{aligned} M_x^{(2)} = &-\frac{(1+\mu)(5-\mu)}{(3+\mu)}\frac{P}{8\pi}\log_e\frac{B_1}{A_1} \\ &+\frac{1-\mu}{3+\mu}\frac{P}{8a}\left[(1+3\mu)v+(1-\mu)y\right]\left(\frac{1}{B_1}-\frac{1}{A_1}\right)\sinh\frac{\pi(y+v)}{a} \\ &+\frac{(1-\mu)^2}{3+\mu}\frac{P\pi vy}{4a^2}\left[\frac{\cosh\dfrac{\pi(v+y)}{a}\cos\dfrac{\pi(x-u)}{a}-1}{B_1^2}\right. \\ &\qquad\left.-\frac{\cosh\dfrac{\pi(v+y)}{a}\cos\dfrac{\pi(x+u)}{a}-1}{A_1^2}\right] \end{aligned}\right\} \tag{81}$$

$$\left.\begin{aligned} M_y^{(2)} &= \frac{(1+\mu)\,P}{8\pi}\log_e\frac{B_1}{A_1} \\ &\quad+\frac{1-\mu}{3+\mu}\,\frac{P}{8a}\left[(3+\mu)v-(1-\mu)y\right]\left(\frac{1}{B_1}-\frac{1}{A_1}\right)\sinh\frac{\pi(y+v)}{a} \\ &\quad-\frac{(1-\mu)^2}{3+\mu}\,\frac{P\pi vy}{4a^2}\left[\frac{\cosh\dfrac{\pi\,(v+y)}{a}\cos\dfrac{\pi\,(x-u)}{a}-1}{B_1^2}\right. \\ &\qquad\left.-\frac{\cosh\dfrac{\pi\,(v+y)}{a}\cos\dfrac{\pi\,(x+u)}{a}-1}{A_1^2}\right]. \end{aligned}\right\}(82)$$

In these equations

$$\left.\begin{matrix} A_1 \\ B_1 \end{matrix}\right\} = \cosh\frac{\pi\,(v+y)}{a} - \cos\frac{\pi\,(x\pm u)}{a}.$$

15. *The Finite Edge Supported by a Flexible Beam.*—Let the slab

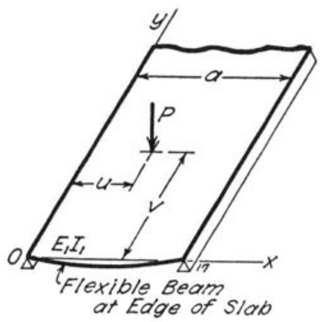



FIG. 17

be as shown in Fig. 17 with the $x$ axis coinciding with a flexible beam which is simply supported at $x = 0$ and at $x = a$.

The boundary conditions at $y = 0$ are

$$\left.\begin{aligned} &\left[\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right]_{y=0} = 0, \\ q = E_1 I_1 \frac{d^4 z}{dx^4} = R_y\Bigg]_{y=0} &= -N\left[\frac{\partial}{\partial y}(\nabla^2 w) + (1-\mu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]_{y=0}, \end{aligned}\right\} \tag{83}$$

where $q$ is the load on the edge beam, positive when downward, and where

$$z = w\Bigg]_{y=0} = \left[w_0 + w_3\right]_{y=0}$$

is the deflection of the beam. Part of the deflection of the slab, $w_0$, is given by (9), and the remainder is given by a correction

$$w_3 = -\frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (c_n + d_n \alpha y)\, e^{-\alpha y} \sin \alpha u \sin \alpha x \tag{84}$$

where $c_n$ and $d_n$ are to be determined by the boundary conditions (83).

From the simultaneous equations in $c_n$ and $d_n$ given by the substitution of $w$ and $z$ into Equations (83), one finds

$$\left.\begin{aligned} c_n &= \frac{2n\pi H_1 (1 + \alpha v) - 4(1 + \mu) - (1 - \mu)^2 (1 + \alpha v)}{2n\pi H_1 + (3 + \mu)(1 - \mu)} e^{-\alpha v}, \\ d_n &= \frac{2n\pi H_1 - (1 - \mu)^2 (1 + 2\alpha v)}{2n\pi H_1 + (3 + \mu)(1 - \mu)} e^{-\alpha v}, \end{aligned}\right\} \tag{85}$$

where
$$H_1 = \frac{E_1 I_1}{aN}.$$

The deflection of the beam is, therefore,

$$z = w\Bigg]_{y=0} = \frac{2Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \left[\frac{2 + (1-\mu)\alpha v}{2n\pi H_1 + (3+\mu)(1-\mu)}\right] e^{-\alpha v} \sin \alpha u \sin \alpha x. \tag{86}$$

The deflection of the slab is given by the sum of (9) and (84) with $c_n$ and $d_n$ given by (85). When $E_1I_1$ approaches $\infty$ the solution reduces to that given in Section 12 under the assumption of a simply supported edge at $y = 0$; when $E_1I_1$ approaches zero the solution reduces to that given in Section 14 under the assumption that the finite edge is free.

16. *Three Edges Simply Supported; Rigid Cross Beam.*—The slab,

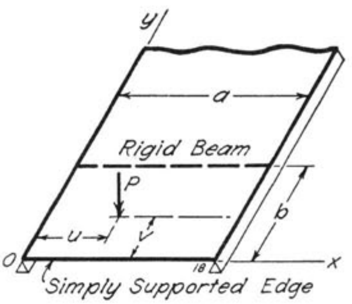



FIG. 18

having the dimensions shown in Fig. 18, has a deflection

$$w = w_0 - w_0' + w_4 \tag{87}$$

where $w_0$ and $w_0'$ are deflections of the infinitely long slab due to loads $P$ at points $(u, v)$ and $(u, -v)$, respectively, and where

$$\left.\begin{aligned} w_4 = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{k}{n^3} \Big\{ & [1 + \alpha\,(y + b)]\, e^{-\alpha(y+b)} \\ & - (1 + \alpha|y-b|)\, e^{-\alpha|y-b|} \Big\} \sin \alpha u \sin \alpha x \end{aligned}\right\} \tag{88}$$

gives the effect of the beam reaction on the slab.

The quantity $k$ is to be determined from the boundary condition

$$w\Big]_{y=b} = 0.$$

One finds

$$k = \frac{(1 + \alpha|b - v|)\, e^{-\alpha|b-v|} - [1 + \alpha\,(b + v)]\, e^{-\alpha(b+v)}}{1 - (1 + 2\alpha b)\, e^{-2\alpha b}} \tag{89}$$

which holds for any position of the load.

When the load is in the end panel, as shown in Fig. 18, $v < b$, and (89) becomes

$$k = \frac{(1 + \alpha b)\sinh \alpha v - \alpha v \cosh \alpha v}{(1 + \alpha b)\sinh \alpha b - \alpha b \cosh \alpha b}. \tag{90}$$

The part of the deflection $w_4$ in the loaded panel, that is, for $0 < y < b$, may then be written in the form

$$w_4 = -\frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{k}{n^3} [(1 + \alpha b) \sinh \alpha y - \alpha y \cosh \alpha y] e^{-\alpha b} \sin \alpha u \sin \alpha x \tag{91}$$

where $k$ is given by (90).

## V. The Rectangular Slab Having Two Opposite Edges Simply Supported and Having Two Edges Stiffened

17. *Concentrated Loads Placed Symmetrically With Respect to the Longitudinal Center Line (Center Line Parallel to the Stiffened Edges).*—The slab and loading are shown in Fig. 19, where a stiffening member

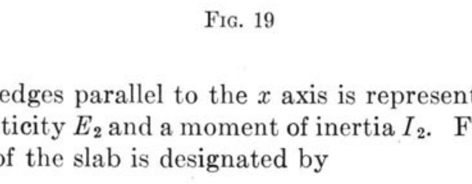



Fig. 19

at each of the edges parallel to the $x$ axis is represented as having a modulus of elasticity $E_2$ and a moment of inertia $I_2$. For convenience, the deflection of the slab is designated by

$$w_1 = w_0' + w' \qquad \text{for} \qquad v > y > -v \tag{92}$$

and

$$w_2 = w_0'' + w' \qquad \text{for} \qquad b > y > v \tag{93}$$

where $w_0'$ and $w_0''$ represent the deflections, in their respective regions, of the infinitely long slab of span $a$ due to the pair of concentrated loads shown in the figure, and where $w'$ represents the deflection due to corrective moments and shears which are applied to the slab along the edges $y = \pm b$ in order to produce the proper edge conditions.

From the solutions given previously, one finds, where $v > y > -v$,

$$w_0' = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \left[(1+\alpha v)\cosh \alpha y - \alpha y \sinh \alpha y\right] e^{-\alpha v} \sin \alpha u \sin \alpha x \tag{94}$$

and, where $b > y > v$,

$$w_0'' = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \left[(1+\alpha y)\cosh \alpha v - \alpha v \sinh \alpha v\right] e^{-\alpha y} \sin \alpha u \sin \alpha x. \tag{95}$$

The deflection due to the symmetrical edge corrections may be written in the form

$$w' = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (b_1 \cosh \alpha y - c_1\, \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x \tag{96}$$

where $b_1$ and $c_1$ are parameters to be determined by the boundary conditions.

The boundary conditions are

$$\left[\frac{\partial^2 w_2}{\partial y^2} + \mu \frac{\partial^2 w_2}{\partial x^2}\right]_{y=b} = 0, \tag{97}$$

$$q = E_2 I_2 \frac{d^4 z}{dx^4} = -R_y\Big]_{y=b} = N\left[\frac{\partial^3 w_2}{\partial y^3} + (2-\mu)\frac{\partial^3 w_2}{\partial x^2 \partial y}\right]_{y=b}, \tag{98}$$

where $q$ is the downward load on the beam and $z$ is the downward deflection. Substitution of the derivatives obtained from (95) and (96) into (97) and (98) results in two equations in $b_1$ and $c_1$ whose simultaneous solution gives

$$b_1 = (f_1 - 1) \cosh \alpha v - (f_3 - 1) \alpha v \sinh \alpha v, \tag{99}$$

$$c_1 = (f_3 - 1) \cosh \alpha v - \frac{2(1-\mu)^2}{\Delta} \alpha v \sinh \alpha v. \tag{100}$$

The quantities $f_1, f_3$ and $\Delta$ are independent of the position of the load and are listed in Appendix B.

The deflection of each of the edge beams is found to be

$$z = w_2 \Big]_{y=b} = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} f_5 \sin \alpha u \sin \alpha x, \tag{101}$$

where the quantity $f_5$ is given in Appendix B. The bending moment in the edge beam is then

$$M_{\text{beam}} = \frac{PaH_2}{\pi} \sum_{1,2,3,\cdots} \frac{1}{n} f_5 \sin \alpha u \sin \alpha x. \tag{102}$$

As the pair of loads traverses the span, $a$, the moment in the edge beam varies until it becomes a maximum at $x = a/2$ when $u = a/2$. The value of this maximum moment is

$$\max.\ M_{\text{beam}} = \frac{PaH_2}{\pi} \sum_{1,3,5,\cdots} \frac{f_5}{n}. \tag{103}$$

The curvatures due to $w'$ are

$$\left.\begin{aligned} \frac{\partial^2 w'}{\partial x^2} &= -\frac{P}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} (b_1 \cosh \alpha y - c_1 \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x, \\ \frac{\partial^2 w'}{\partial y^2} &= \frac{P}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} [(b_1 - 2c_1) \cosh \alpha y - c_1 \alpha y \sinh \alpha y] \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{104}$$

At $y = 0$ these become

$$\left.\begin{aligned} \frac{\partial^2 w'}{\partial x^2} \Big]_{y=0} &= -\frac{P}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} b_1 \sin \alpha u \sin \alpha x, \\ \frac{\partial^2 w'}{\partial y^2} \Big]_{y=0} &= \frac{P}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} (b_1 - 2c_1) \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{105}$$

When the loads are within the central region of the slab, that is within the middle quarter of the span in either the $x$ or $y$ direction, Equations (105), evaluated for $x = u = a/2$, may be used for an approximate computation of the corrective moments under the loads. The approximation is possible because the correction is a symmetrical edge effect which varies but little over the central area of the slab.

18. *Concentrated Load on the Longitudinal Center Line.*—When the

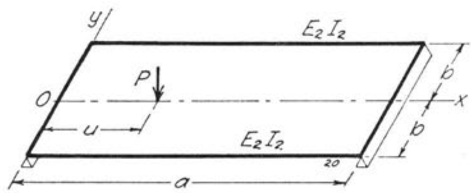



FIG. 20

load is on the longitudinal center line, as shown in Fig. 20, the deflection of the slab for $b > y > 0$ is

$$w = w_0 + w' \tag{106}$$

where

$$w_0 = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (1 + \alpha y) e^{-\alpha y} \sin \alpha u \sin \alpha x \tag{107}$$

and

$$w' = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} [(f_1 - 1) \cosh \alpha y - (f_3 - 1) \alpha y \sinh \alpha y] \sin \alpha u \sin \alpha x. \tag{108}$$

The complete deflection of the slab for $b > y > 0$ is

$$\left.\begin{aligned} w = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (f_1 \cosh \alpha y &- \sinh \alpha y \\ &+ \alpha y \cosh \alpha y - f_3 \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{109}$$

The corrective curvatures on the line $y = 0$, due to $w'$, are

$$\left.\begin{aligned} \left.\frac{\partial^2 w'}{\partial x^2}\right]_{y=0} &= -\frac{P}{2\pi N}\sum_{1,2,3,\cdots}\frac{1}{n}(f_1 - 1)\sin\alpha u\sin\alpha x, \\ \left.\frac{\partial^2 w'}{\partial y^2}\right]_{y=0} &= \frac{P}{2\pi N}\sum_{1,2,3,\cdots}\frac{1}{n}(f_1 - 2f_3 + 1)\sin\alpha u\sin\alpha x. \end{aligned}\right\} \quad (110)$$

The deflection of each edge beam is

$$z = w\Big]_{y=b} = \frac{Pa^2}{2\pi^3 N}\sum_{1,2,3,\cdots}\frac{1}{n^3}f_2\sin\alpha u\sin\alpha x, \quad (111)$$

where $f_2$, stated in Appendix B, is a function which is dependent on the ratio $b/a$ and upon the relative stiffness of the edge beams and the slab. The moment in each of the beams is

$$M_{\text{beam}} = \frac{PaH_2}{2\pi}\sum_{1,2,3,\cdots}\frac{1}{n}f_2\sin\alpha u\sin\alpha x. \quad (112)$$

19. *Concentrated Loads Over the Stiffened Edges; Loads Symmetrical With Respect to the Longitudinal Center Line.*—When $v = b$ in the solution given in Section 17 the loads are over the stiffened edges

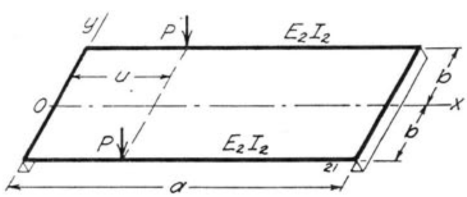



FIG. 21

as shown in Fig. 21. The total deflection of the slab is then given by the equation

$$w = \frac{Pa^2}{\pi^3 N}\sum_{1,2,3,\cdots}\frac{1}{n^3}(f_2\cosh\alpha y - f_4\,\alpha y\sinh\alpha y)\sin\alpha u\sin\alpha x. (113)$$

The quantities $f_2$ and $f_4$ are given in Appendix B. Moments computed from (113) are finite within the entire slab provided that $E_2$ and $I_2$, the modulus of elasticity and moment of inertia of the edge beams, are not zero.

The deflection of each of the edge beams becomes

$$\left.\begin{aligned} z = w\Big]_{y=b} &= \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} f_7 \sin \alpha u \sin \alpha x \\ &= z_0 - \frac{2Pa^3}{\pi^4 E_2 I_2} \sum_{1,2,3,\cdots} \frac{1}{n^4} f_8 \sin \alpha u \sin \alpha x \end{aligned}\right\} \tag{114}$$

where

$$\left.\begin{aligned} z_0 &= \frac{2Pa^3}{\pi^4 E_2 I_2} \sum_{1,2,3,\cdots} \frac{1}{n^4} \sin \alpha u \sin \alpha x \\ &= \frac{P(a-u)x}{6E_2 I_2 a}(2au - x^2 - u^2) \qquad \text{for} \quad x < u \\ &= \frac{P(a-x)u}{6E_2 I_2 a}(2ax - u^2 - x^2) \qquad \text{for} \quad x > u. \end{aligned}\right\} \tag{115}$$

Here $z_0$ is the simple beam deflection for the edge beam supporting the concentrated load, but without the effect of the slab. The quantities $f_7$ and $f_8$ are given in Appendix B.

The bending moment in either edge beam is

$$M_{\text{beam}} = -E_2 I_2 \frac{d^2 z_0}{dx^2} - \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} f_8 \sin \alpha u \sin \alpha x.$$

When the loads are at the center of the edge beams, the moment under each load becomes

$$\text{max. } M_{\text{beam}} = \frac{Pa}{4}\left(1 - \frac{8}{\pi^2} \sum_{1,3,5,\cdots} \frac{f_8}{n^2}\right).$$

At the edge of the slab the bending moment $M_y$ is zero and

$$M_x\Big]_{y=b} = \frac{1-\mu^2}{aH_2} M_{\text{beam}}. \tag{116}$$

20. *Concentrated Loads on the Slab Giving a Slab Deflection Which Is Anti-symmetrical With Respect to the Longitudinal Center Line.*— The slab is shown in Fig. 22. The part of the slab in the region of positive $y$ behaves as if it were a rectangular slab of width $b$ and length $a$, simply supported on the three edges $x = 0$, $x = a$, $y = 0$, supported by an edge beam at $y = b$, and loaded by a single con-

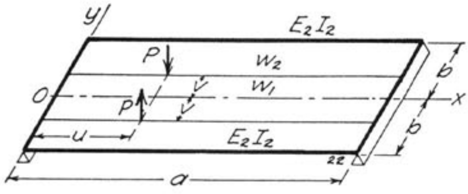



Fig. 22

centrated load $P$ at the point $x = u$, $y = v$. For the slab as shown in Fig. 22 the presence of a supporting beam along the line $y = 0$ has no effect upon the solution since there is no deflection along that line.

One may divide the slab into regions and express the deflection of the slab in these regions by the equations

$$w_1 = w_0' + w' \qquad \text{for} \qquad v > y > -v \tag{117}$$

and

$$w_2 = w_0'' + w' \qquad \text{for} \qquad b > y > v. \tag{118}$$

In these equations $w_0'$ and $w_0''$ are deflections of the infinitely long slab of span $a$ due to the pair of anti-symmetrical loads shown in Fig. 22, and $w'$ is the deflection due to corrections applied at the edges $y = \pm b$. Thus:

$$w_0' = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} [(1+\alpha v) \sinh \alpha y - \alpha y \cosh \alpha y]\, e^{-\alpha v} \sin \alpha u \sin \alpha x, \tag{119}$$

$$w_0'' = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} [(1+\alpha y) \sinh \alpha v - \alpha v \cosh \alpha v]\, e^{-\alpha y} \sin \alpha u \sin \alpha x, \tag{120}$$

$$w' = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (a_1 \sinh \alpha y + d_1\, \alpha y \cosh \alpha y) \sin \alpha u \sin \alpha x. \tag{121}$$

From boundary conditions identical with those expressed by Equations (97) and (98), one finds

$$a_1 = (F_3 - 1)\ \alpha v \cosh \alpha v - (F_1 - 1) \sinh \alpha v, \tag{122}$$

$$d_1 = (F_3 - 1) \sinh \alpha v + \frac{2\,(1 - \mu)^2}{\Delta_1}\ \alpha v \cosh \alpha v, \tag{123}$$

where $F_1$, $F_3$ and $\Delta_1$ are quantities given in Appendix B.

The deflection of the edge beam at $y = b$ is

$$z = w_2\Big]_{y=b} = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3}\ F_5 \sin \alpha u \sin \alpha x \tag{124}$$

where the quantity $F_5$ is given in Appendix B. The bending moment in the edge beam is

$$M_{\text{beam}} = \frac{PaH_2}{\pi} \sum_{1,2,3,\cdots} \frac{1}{n}\ F_5 \sin \alpha u \sin \alpha x. \tag{125}$$

The curvatures due to $w'$ are

$$\left.\begin{aligned} \frac{\partial^2 w'}{\partial x^2} &= -\frac{P}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n}\ (a_1 \sinh \alpha y + d_1\ \alpha y \cosh \alpha y) \sin \alpha u \sin \alpha x, \\ \frac{\partial^2 w'}{\partial y^2} &= \frac{P}{\pi N} \sum_{1,2,3,\cdots} \frac{1}{n} [(a_1 + 2d_1) \sinh \alpha y + d_1\ \alpha y \cosh \alpha y] \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{126}$$

21. *Concentrated Loads Over the Stiffened Edges Giving a Slab Deflection Which Is Anti-symmetrical With Respect to the Longitudinal Center Line.*—When $v = b$ in the preceding solution the loads are

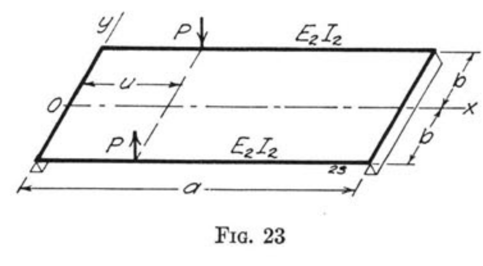



FIG. 23

over the stiffened edges and the total deflection, valid over the entire area of slab, becomes

$$w = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (F_2 \sinh \alpha y - F_4\, \alpha y \cosh \alpha y) \sin \alpha u \sin \alpha x \quad (127)$$

where $F_2$ and $F_4$ are quantities given in Appendix B.

The deflection of the beam at $y = b$ becomes

$$\left.\begin{aligned} z_2 = w\Big]_{y=b} &= \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} F_7 \sin \alpha u \sin \alpha x \\ &= z_0 - \frac{2Pa^3}{\pi^4 E_2 I_2} \sum_{1,2,3,\cdots} \frac{1}{n^4} F_8 \sin \alpha u \sin \alpha x, \end{aligned}\right\} \quad (128)$$

where $z_0$ is the simple beam deflection given by (115) and where $F_7$ and $F_8$ are functions stated in Appendix B.

The bending moment in the beam at $y = b$ is

$$M_{\text{beam}} = -E_2 I_2 \frac{d^2 z_0}{dx^2} - \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} F_8 \sin \alpha u \sin \alpha x. \quad (129)$$

This moment is a maximum under the load when $u = a/2$. Then

$$\text{max. } M_{\text{beam}} = \frac{Pa}{4} \left( 1 - \frac{8}{\pi^2} \sum_{1,3,5,\cdots} \frac{F_8}{n^2} \right). \quad (130)$$

As before, the bending moment $M_x$ at the edge of the slab may be found from the moment in the beam by the relation

$$M_x\Big]_{y=b} = \frac{1 - \mu^2}{aH_2} M_{\text{beam}}.$$

22. *Load Uniformly Distributed Over the Entire Slab.**—When the intensity of load is $p$ per unit of area and the axes are chosen as shown

*This problem has been treated by others. See, for example, Emil Müller, "Über rechteckige Platten, die längs zweier gegenüberliegenden Seiten auf biegsamen Trägern ruhen," Zeit. für angew. Math. und Mech., 6, 1926, p. 355-66. Müller gives numerical values of bending moments at the center of the slab, assuming that Poisson's ratio $\mu = 0$, for ratios of $b/a$ varying from 1/8 to 1.0, and for ratios of $E_1 I_1/(bN)$ of 0, 0.5, 1.0, 1.5, 2.0, 2.5 and 3.0. See also B. G. Galerkin, "Elastic Thin Plates," Gosstrojisdat, Leningrad and Moscow, 1933, (In Russian) p. 86.

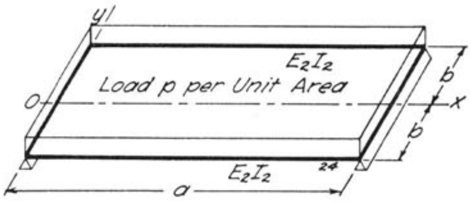



FIG. 24

in Fig. 24, the deflection $w$ may be taken as the sum of the deflection of a uniformly loaded beam,

$$w' = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} \sin \alpha x = \frac{p}{24N} (x^4 - 2ax^3 + a^3 x), \quad (131)$$

and a correction

$$w'' = -\frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} (b_3 \cosh \alpha y - c_3\, \alpha y \sinh \alpha y) \sin \alpha x. \quad (132)$$

Thus, for any value of $y$ within the slab,

$$w = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} (1 - b_3 \cosh \alpha y + c_3\, \alpha y \sinh \alpha y) \sin \alpha x. \quad (133)$$

The boundary conditions at $y = b$, similar to (97) and (98), when applied to $w$, give the results:

$$\left.\begin{aligned} \Delta b_3 &= 2n\pi H_2 (2 \cosh \alpha b + \alpha b \sinh \alpha b) \\ &\qquad + 2\mu \left[(1 - \mu)\, \alpha b \cosh \alpha b - (1 + \mu) \sinh \alpha b\right], \\ \Delta c_3 &= 2n\pi H_2 \cosh \alpha b + 2\mu (1 - \mu) \sinh \alpha b, \end{aligned}\right\} \quad (134)$$

where $\Delta$ is given in Appendix B. In the special case when $H_2 = 0$ this solution reduces to that given by Holl.* In the other limiting case, when $H_2 = \infty$, the solution reduces to that given by Nádai† for the uniformly loaded rectangular slab simply supported on four sides.

*D. L. Holl, "Analysis of Thin Rectangular Plates Supported on Opposite Edges," Bulletin 129, Iowa Engineering Experiment Station, Iowa State College, 1936, pp. 15, 87, and 88.

†A. Nádai, Die elastischen Platten, 1925, p. 123.

The deflection of each edge beam is

$$z = w \Big]_{y=b} = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} d_3 \sin \alpha x \qquad (135)$$

where

$$\Delta d_3 = (3 - \mu) \sinh 2\alpha b - 2(1 - \mu) \alpha b. \qquad (136)$$

The curvatures at the center of the slab are

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial x^2}\Big]_{\substack{x=a/2\\y=0}} &= -\frac{pa^2}{8N}\left(1 - \frac{32}{\pi^3}\sum_{1,3,5,\cdots}\frac{(-1)^{\frac{n-1}{2}}}{n^3} b_3\right), \\ \frac{\partial^2 w}{\partial y^2}\Big]_{\substack{x=a/2\\y=0}} &= -\frac{pa^2}{8N}\frac{32}{\pi^3}\sum_{1,3,5,\cdots}\frac{(-1)^{\frac{n-1}{2}}}{n^3}(b_3 - 2c_3). \end{aligned}\right\} \qquad (137)$$

From these, the moments at the center of the slab are readily determined.

The maximum bending moment at the center of each edge beam is

$$M_{\text{beam}}\Big]_{x=a/2} = \frac{4pa^3 H_2}{\pi^3} \sum_{1,3,5,\cdots} \frac{(-1)^{\frac{n-1}{2}}}{n^3} d_3. \qquad (138)$$

The bending moment $M_x$ in the slab immediately above an edge beam may be obtained from the corresponding moment in the beam by means of the relation given by (116).

23. *Uniform Line Loads on the Stiffened Edges.*—If the load per unit of length along the edges $y = \pm b$, as shown in Fig. 25, is

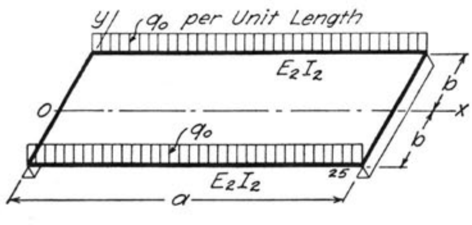



FIG. 25

designated by $q_0$, the deflection of the slab may be found from (113) by replacing $P$ by $(q_0\, du)$ and then integrating with respect to $u$. One finds the deflection of the slab to be

$$w = \frac{2q_0a^3}{\pi^4 N} \sum_{1,3,5,\cdots} \frac{1}{n^4} (f_2 \cosh \alpha y - f_4\, \alpha y \sinh \alpha y) \sin \alpha x. \quad (139)$$

The deflection of either edge beam is

$$\left.\begin{aligned} z = w\Big]_{y=b} &= \frac{2q_0a^3}{\pi^4 N} \sum_{1,3,5,\cdots} \frac{1}{n^4} f_7 \sin \alpha x \\ &= \frac{4q_0a^4}{\pi^5 E_2 I_2} \sum_{1,3,5,\cdots} \frac{1}{n^5} (1 - f_8) \sin \alpha x \\ &= \frac{q_0 x}{24\, E_2 I_2} (a^3 - 2ax^2 + x^3) - \frac{4q_0a^4}{\pi^5 E_2 I_2} \sum_{1,3,5,\cdots} \frac{1}{n^5} f_8 \sin \alpha x \end{aligned}\right\} (140)$$

where

$$\Delta f_8 = (3 + \mu)(1 - \mu) \sinh 2\alpha b - 2(1 - \mu)^2\, \alpha b. \quad (141)$$

The curvatures of the slab at the center are

$$\left.\begin{aligned} \frac{\partial^2 w}{\partial x^2}\Big]_{\substack{x=a/2\\ y=0}} &= -\frac{2q_0 a}{\pi^2 N} \sum_{1,3,5,\cdots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} f_2, \\ \frac{\partial^2 w}{\partial y^2}\Big]_{\substack{x=a/2\\ y=0}} &= \frac{2q_0 a}{\pi^2 N} \sum_{1,3,5,\cdots} \frac{(-1)^{\frac{n-1}{2}}}{n^2} (f_2 - 2f_4). \end{aligned}\right\} (142)$$

The bending moment in either edge beam is

$$\left.\begin{aligned} M_{\text{beam}} &= \frac{2q_0a^2 H_2}{\pi^2} \sum_{1,3,5,\cdots} \frac{1}{n^2} f_7 \sin \alpha x \\ &= \frac{q_0 x}{2} (a - x) - \frac{4q_0 a^2}{\pi^3} \sum_{1,3,5,\cdots} \frac{1}{n^3} f_8 \sin \alpha x. \end{aligned}\right\} (143)$$

The maximum moment at the center of the beam is

$$\text{max. } M_{\text{beam}} = \frac{q_0 a^2}{8}\left(1 - \frac{32}{\pi^3} \sum_{1,3,5,\cdots} \frac{(-1)^{\frac{n-1}{2}}}{n^3} f_8\right). \tag{144}$$

## VI. The Rectangular Slab Having Two Opposite Edges Simply Supported and Having Edge and Center Stringers

24. *Concentrated Loads Placed Symmetrically With Respect to the Center Stringer.*—The slab and loading are shown in Fig. 26. From

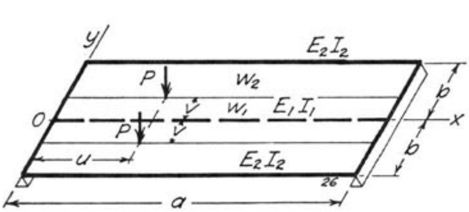



Fig. 26

the results given in Sections 17 and 18 one may obtain the deflection of the slab,

$$w_1 = w_0' + w' + \overline{w} \qquad \text{for} \quad v > y > 0 \tag{145}$$

and

$$w_2 = w_0'' + w' + \overline{w} \qquad \text{for} \quad b > y > v, \tag{146}$$

where $w_0'$, $w_0''$ and $w'$ are given by Equations (94), (95) and (96) respectively, and where

$$\left.\begin{aligned}\overline{w} = -\frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{k_1}{n^3} (f_1 \cosh \alpha y - \sinh \alpha y \\ + \alpha y \cosh \alpha y - f_3\, \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x.\end{aligned}\right\} \tag{147}$$

Equation (147) represents the effect of the center beam, and is similar to (109) except for the load distribution factor, $k_1$, which is to be determined from the boundary condition at $y = 0$. The quantities $f_1$ and $f_3$ are given in Appendix B.

If the load on the center beam is designated by $q_1$ and the deflection is designated by $z_1$, both positive when downward, the boundary condition becomes

$$q_1 = E_1 I_1 \frac{d^4 z_1}{dx^4} = -2N \left[ \frac{\partial}{\partial y} (\nabla^2 \overline{w}) \right]_{y=0} . \tag{148}$$

From this condition, one finds

$$k_1 = \frac{b_1 + (1 + \alpha v)\, e^{-\alpha v}}{\dfrac{4}{n\pi H_1} + f_1}. \tag{149}$$

The deflection of the center beam may then be expressed by the equation

$$z_1 = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \frac{4}{n\pi H_1} k_1 \sin \alpha u \sin \alpha x. \tag{150}$$

The deflection of each of the outer beams is found to be

$$z_2 = w_2 \Big]_{y=b} = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (f_5 - k_1 f_2) \sin \alpha u \sin \alpha x \tag{151}$$

where the quantities $f_2$ and $f_5$ are given in Appendix B.

The deflection of the slab for $y > 0$ has been expressed by Equations (145) and (146) as the sum of two parts: [1] the deflection of the infinitely long slab of span $a$ due to the pair of concentrated loads shown in Fig. 26, and [2] a correction $w_{cor} = w' + \overline{w}$. Thus:

$$\left.\begin{aligned} w_{cor} = {} & \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (b_1 \cosh \alpha y - c_1\, \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x \\ & - \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{k_1}{n^3} (f_1 \cosh \alpha y - \sinh \alpha y + \alpha y \cosh \alpha y \\ & \qquad - f_3 \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x. \end{aligned}\right\} \tag{152}$$

The quantities $b_1$, $c_1$, $k_1$, $f_1$ and $f_3$ are stated in Appendix B.

25. *Concentrated Load Over the Center Stringer.*—By letting $v = 0$ in the preceding solution one obtains the solution for the special case

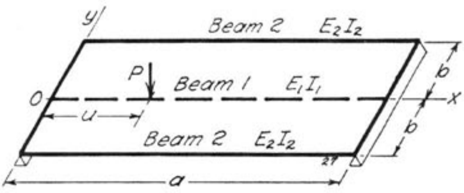



FIG. 27

in which the load is over the center stringer as shown in Fig. 27. The total deflection of the slab for $y > 0$ is found to be

$$\left. \begin{aligned} w = \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2}{n^4} (f_1 \cosh \alpha y - \sinh \alpha y \\ + \alpha y \cosh \alpha y - f_3 \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x \end{aligned} \right\} \quad (153)$$

where

$$k_2 = \frac{1}{\dfrac{4}{n\pi H_1} + f_1} \quad (154)$$

and where $f_1$ and $f_3$ are functions given in Appendix B.

The deflection of the center beam is

$$\left. \begin{aligned} z_1 = w \Big]_{y=0} &= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2 f_1}{n^4} \sin \alpha u \sin \alpha x \\ &= z_0 - \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4} \frac{4}{n\pi H_1} k_2 \sin \alpha u \sin \alpha x \end{aligned} \right\} \quad (155)$$

where $z_0$ is the deflection of the center beam supporting the concentrated load, but without the restraining effect of the slab; that is

$$\left. \begin{aligned} z_0 &= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4} \sin \alpha u \sin \alpha x \\ &= \frac{P(a-u)x}{6E_1 I_1 a} (2au - x^2 - u^2) \qquad \text{for} \qquad x < u \\ &= \frac{P(a-x)u}{6E_1 I_1 a} (2ax - u^2 - x^2) \qquad \text{for} \qquad x > u. \end{aligned} \right\} \quad (156)$$

The deflection of either edge beam is

$$z_2 = w\Big]_{y=b} = \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2 f_2}{n^4} \sin \alpha u \sin \alpha x. \tag{157}$$

The curvatures of the slab over the center beam may be found by differentiation of (155) and (153). One finds, at $y = 0$:

$$\left.\begin{aligned}
\frac{\partial^2 w}{\partial x^2}\Big]_{y=0} &= \frac{d^2 z_0}{dx^2} + \frac{2Pa}{\pi^2 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^2} \frac{4}{n\pi H_1} k_2 \sin \alpha u \sin \alpha x, \\
\frac{\partial^2 w}{\partial y^2}\Big]_{y=0} &= \frac{2Pa}{\pi^2 E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2}{n^2} (f_1 - 2f_3) \sin \alpha u \sin \alpha x \\
&= \frac{\partial^2 w}{\partial x^2}\Big]_{y=0} + \frac{4Pa}{\pi^2 E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2}{n^2} (f_1 - f_3) \sin \alpha u \sin \alpha x.
\end{aligned}\right\} \tag{158}$$

The bending moment in the center beam is

$$M_{\text{beam1}} = -E_1 I_1 \frac{d^2 z_0}{dx^2} - \frac{2Pa}{\pi^2} \sum_{1,2,3,\cdots} \frac{1}{n^2} \frac{4}{n\pi H_1} k_2 \sin \alpha u \sin \alpha x \tag{159}$$

and the moment in each of the outer beams is

$$M_{\text{beam2}} = \frac{2Pa}{\pi^2} \frac{E_2 I_2}{E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2 f_2}{n^2} \sin \alpha u \sin \alpha x. \tag{160}$$

When the load is at the center of Beam 1 the bending moment in that beam, under the load, is

$$\text{max. } M_{\text{beam1}} = \frac{Pa}{4}\left(1 - \frac{8}{\pi^2} \sum_{1,3,5,\cdots} \frac{1}{n^2} \frac{4}{n\pi H_1} k_2\right). \tag{161}$$

26. *Concentrated Loads Over the Edge Stringers; Loads Symmetrical With Respect to the Center Stringer.*—The slab is shown in Fig. 28.

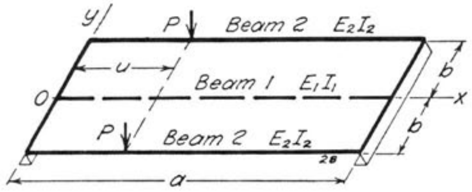



FIG. 28

From previous results the slab deflection for $b > y > 0$ is found to be

$$w = \frac{Pa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{k_2}{n^3} \left[ f_2 \left( \frac{4}{n\pi H_1} \cosh \alpha y + \sinh \alpha y - \alpha y \cosh \alpha y \right) + \frac{4}{n\pi H_1} f_9 \, \alpha y \sinh \alpha y \right] \sin \alpha u \sin \alpha x \quad (162)$$

where $f_2$ and $f_9$ are given in Appendix B, and where

$$k_2 = \frac{1}{\dfrac{4}{n\pi H_1} + f_1}. \quad (163)$$

The deflection of the center beam is

$$z_1 = w \Big]_{y=0} = \frac{4Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{k_2}{n^4} f_2 \sin \alpha u \sin \alpha x \quad (164)$$

and the deflection of each of the outer beams is

$$\begin{aligned} z_2 = w \Big]_{y=b} &= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,2,3,\cdots} \frac{1}{n^4} k_2 f_6 \sin \alpha u \sin \alpha x \\ &= z_0 - \frac{2Pa^3}{\pi^4 E_2 I_2} \sum_{1,2,3,\cdots} \frac{1}{n^4} \left( 1 - \frac{E_2 I_2}{E_1 I_1} k_2 f_6 \right) \sin \alpha u \sin \alpha x \end{aligned} \quad (165)$$

where $z_0$ is the simple beam deflection given by (115), and where $f_6$ is given in Appendix B.

27. *Concentrated Loads Giving a Slab Deflection Which Is Anti-symmetrical With Respect to the Longitudinal Center Line.*—The neglect of the effect of twisting of the beams and the fact that there is no deflection of the longitudinal center line permit the corresponding solutions given in Chapter V, Sections 20 and 21 to be used also when the slab is continuous over a longitudinal center stringer. These anti-symmetrical solutions may be combined with the corresponding symmetrical solutions to obtain the effect of a single concentrated load at any point on the slab.

28. *Load Uniformly Distributed Over the Entire Slab.**—Section 22 gives the solution for the uniformly loaded slab with edge stiffeners only, namely

$$w_1 = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} (1 - b_3 \cosh \alpha y + c_3 \alpha y \sinh \alpha y) \sin \alpha x \quad (166)$$

where $b_3$ and $c_3$ are given in Appendix B. To this a second solution,

$$\left. \begin{aligned} w_2 = -\frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{k_3}{n^5} (f_1 \cosh \alpha y - \sinh \alpha y \\ + \alpha y \cosh \alpha y - f_3 \alpha y \sinh \alpha y) \sin \alpha x, \end{aligned} \right\} (167)$$

similar to (109), may be added to correct for the effect of the center

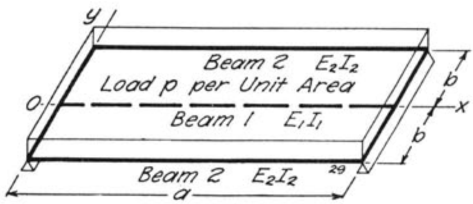



FIG. 29

*This problem has been treated previously in an approximate manner. See Hjalmar Granholm, "Om Lastfördelning Mellan Balkar Med Särskild Hänsyn Till Farbaneplattans Inverkan," Betong, 1936, No. 4, pp. 139-219, Stockholm. Granholm takes account of the interaction between the beams and slab and also includes the effect of torsional rigidity of the beams in his analyses, but considers the slab to be replaced by a series of thin strips which act independently of each other.

beam. The resulting expression, $w = w_1 + w_2$, is applicable when $y > 0$ in the slab shown in Fig. 29.

The boundary condition, for determining $k_3$, is

$$q_1 = E_1 I_1 \frac{d^4 z_1}{dx^4} = -2N \left[ \frac{\partial}{\partial y} (\nabla^2 w_2) \right]_{y=0} \tag{168}$$

where $z_1$ is the deflection of the center beam. From (168) one finds

$$k_3 = \frac{1 - b_3}{\dfrac{4}{n\pi H_1} + f_1} = (1 - b_3)\, k_2 \tag{169}$$

where $k_2$ is given by (163) and where $f_1$ and $b_3$ are given in Appendix B.

The beam deflections are

$$z_1 = w \bigg]_{y=0} = \frac{16pa^5}{\pi^6 E_1 I_1} \sum_{1,3,5,\cdots} \frac{1}{n^6}\, k_3 \sin \alpha x \tag{170}$$

for the center beam, and

$$z_2 = w \bigg]_{y=b} = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} (d_3 - k_3 f_2) \sin \alpha x \tag{171}$$

for the edge beams, where $d_3$ and $f_2$ are given in Appendix B and where $k_3$ is given by (169).

The bending moments in the beams are

$$M_{\text{beam}1} = \frac{16pa^3}{\pi^4} \sum_{1,3,5,\cdots} \frac{k_3}{n^4} \sin \alpha x \tag{172}$$

for the center beam, and

$$M_{\text{beam}2} = \frac{4pa^3 H_2}{\pi^3} \sum_{1,3,5,\cdots} \frac{1}{n^3} (d_3 - k_3 f_2) \sin \alpha x \tag{173}$$

for the edge beams.

## VII. Examples of Special Solutions Obtainable From Those Given in Previous Chapters

29. *Rectangular Slab Having Two Opposite Edges Simply Supported and Two Edges Supported on Beams Which Are Continuous Over Interior Supports; Concentrated Load on the Longitudinal Center Line.*—The slab is shown in Fig. 30. If the concentrated interior

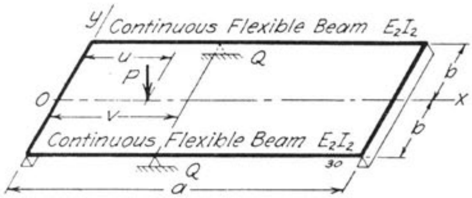



Fig. 30

reactions are designated by $Q$, one may write an equation for the deflection of the slab, for $y > 0$:

$$\left.\begin{aligned} w = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (f_1 \cosh \alpha y - \sinh \alpha y \\ + \alpha y \cosh \alpha y - f_3\, \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x \\ - \frac{Qa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (f_2 \cosh \alpha y - f_4\, \alpha y \sinh \alpha y) \sin \alpha v \sin \alpha x. \end{aligned}\right\} \tag{174}$$

This equation is simply the sum of two equations of the form of (109) and (113), respectively.

The deflection of the continuous edge beam is

$$\left.\begin{aligned} z = w\Big]_{y=b} = \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} f_2 \sin \alpha u \sin \alpha x \\ - \frac{Qa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} f_7 \sin \alpha v \sin \alpha x. \end{aligned}\right\} \tag{175}$$

The requirement of zero deflection at the interior supports gives

$$w\Big]_{\substack{x=v\\y=b}} = 0, \tag{176}$$

from which

$$Q = \frac{P}{2}\frac{\sum\limits_{1,2,3,\cdots}\dfrac{1}{n^3}f_2 \sin\alpha u \sin\alpha v}{\sum\limits_{1,2,3,\cdots}\dfrac{1}{n^3}f_7 \sin^2\alpha v}. \tag{177}$$

The quantities $f_1$, $f_2$, $f_3$, $f_4$ and $f_7$ in the foregoing equations are given in Appendix B.

30. *Rectangular Slab Having Two Opposite Edges Simply Supported and Two Edges Supported on Beams Which Are Continuous Over Interior Supports; Load Uniformly Distributed Over the Entire Slab.*—The slab and axes are shown in Fig. 31 where the intensity

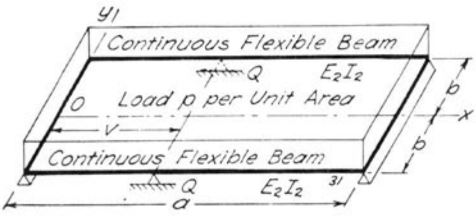



FIG. 31

of the uniformly distributed load is indicated as $p$ per unit of area. Combining equations of the type of (133) and (113), one obtains the deflection of the slab

$$\left.\begin{aligned} w = {} & \frac{4pa^4}{\pi^5 N}\sum_{1,3,5,\cdots}\frac{1}{n^5}(1 - b_3 \cosh\alpha y + c_3\,\alpha y \sinh\alpha y)\sin\alpha x \\ & - \frac{Qa^2}{\pi^3 N}\sum_{1,2,3,\cdots}\frac{1}{n^3}(f_2 \cosh\alpha y - f_4\,\alpha y \sinh\alpha y)\sin\alpha v \sin\alpha x. \end{aligned}\right\} \tag{178}$$

The deflection of the continuous edge beam is

$$z = w\Big]_{y=b} = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} d_3 \sin \alpha x - \frac{Qa^2}{\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} f_7 \sin \alpha v \sin \alpha x. \quad (179)$$

One finds, for zero deflection at the interior supports, that the concentrated reactions are given by the equation

$$Q = \frac{4pa^2}{\pi^2} \frac{\displaystyle\sum_{1,3,5,\cdots} \frac{1}{n^3} d_3 \sin \alpha v}{\displaystyle\sum_{1,2,3,\cdots} \frac{1}{n^3} f_7 \sin^2 \alpha v}. \quad (180)$$

The functions $b_3$, $c_3$, $d_3$, $f_2$, $f_4$ and $f_7$ are given in Appendix B.

31. *Rectangular Slab Having Two Opposite Edges Simply Supported and Two Edges Supported on Beams Which Are Continuous Over Interior Supports at the Third-Points; Concentrated Load on the Longitudinal Center Line.*—The slab is shown in Fig. 32 where each

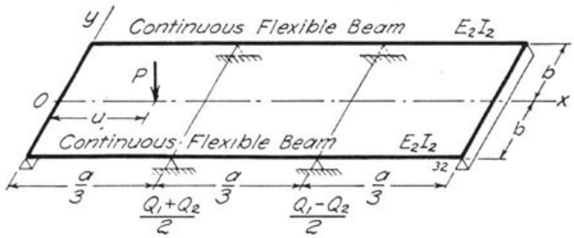



FIG. 32

of the interior reactions at $x = a/3$ is represented as having a magnitude of $(Q_1 + Q_2)/2$ and the remaining interior reactions are each represented as having a magnitude of $(Q_1 - Q_2)/2$. From (109) and (113) one obtains a deflection function for $b > y > 0$:

$$\left.\begin{aligned} w = {} & \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} (f_1 \cosh \alpha y - \sinh \alpha y \\ & \qquad + \alpha y \cosh \alpha y - f_3 \, \alpha y \sinh \alpha y) \sin \alpha u \sin \alpha x \\ & - \frac{Q_1 a^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} (f_2 \cosh \alpha y - f_4 \, \alpha y \sinh \alpha y) \sin \frac{n\pi}{3} \sin \alpha x \\ & - \frac{Q_2 a^2}{\pi^3 N} \sum_{2,4,6,\cdots} \frac{1}{n^3} (f_2 \cosh \alpha y - f_4 \, \alpha y \sinh \alpha y) \sin \frac{n\pi}{3} \sin \alpha x. \end{aligned}\right\} \quad (181)$$

The deflection of each of the edge beams is then

$$\left.\begin{aligned} z = w \Big]_{y=b} = {} & \frac{Pa^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} f_2 \sin \alpha u \sin \alpha x \\ & - \frac{Q_1 a^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} f_7 \sin \frac{n\pi}{3} \sin \alpha x \\ & - \frac{Q_2 a^2}{\pi^3 N} \sum_{2,4,6,\cdots} \frac{1}{n^3} f_7 \sin \frac{n\pi}{3} \sin \alpha x. \end{aligned}\right\} \quad (182)$$

The conditions

$$w \Big]_{\substack{y=b \\ x=a/3}} = w \Big]_{\substack{y=b \\ x=2a/3}} = 0$$

give the relations

$$\left.\begin{aligned} Q_1 &= \frac{P}{2} \frac{\displaystyle\sum_{1,3,5,\cdots} \frac{1}{n^3} f_2 \sin \alpha u \sin \frac{n\pi}{3}}{\displaystyle\sum_{1,3,5,\cdots} \frac{1}{n^3} f_7 \sin^2 \frac{n\pi}{3}} \\ Q_2 &= \frac{P}{2} \frac{\displaystyle\sum_{2,4,6,\cdots} \frac{1}{n^3} f_2 \sin \alpha u \sin \frac{n\pi}{3}}{\displaystyle\sum_{2,4,6,\cdots} \frac{1}{n^3} f_7 \sin^2 \frac{n\pi}{3}}. \end{aligned}\right\} \quad (183)$$

The quantities $f_1$, $f_2$, $f_3$, $f_4$ and $f_7$ are given in Appendix B.

32. *Rectangular Slab Having Two Opposite Edges Simply Supported and Two Edges Supported on Beams Which Are Continuous Over Interior Supports at the Third-Points; Load Uniformly Distributed*

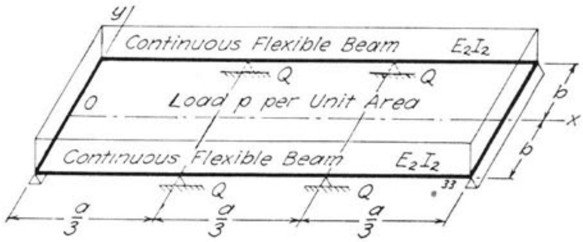



FIG. 33

*Over the Entire Slab.*—For the entire slab shown in Fig. 33 the deflection is given by the equation

$$\left.\begin{aligned} w = {} & \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} (1 - b_3 \cosh \alpha y + c_3 \alpha y \sinh \alpha y) \sin \alpha x \\ & - \frac{2Qa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} (f_2 \cosh \alpha y - f_4 \alpha y \sinh \alpha y) \sin \frac{n\pi}{3} \sin \alpha x. \end{aligned}\right\} \quad (184)$$

The deflection of each of the edge beams is

$$z = w\Big]_{y=b} = \frac{4pa^4}{\pi^5 N} \sum_{1,3,5,\cdots} \frac{1}{n^5} d_3 \sin \alpha x - \frac{2Qa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} f_7 \sin \frac{n\pi}{3} \sin \alpha x. \quad (185)$$

The condition

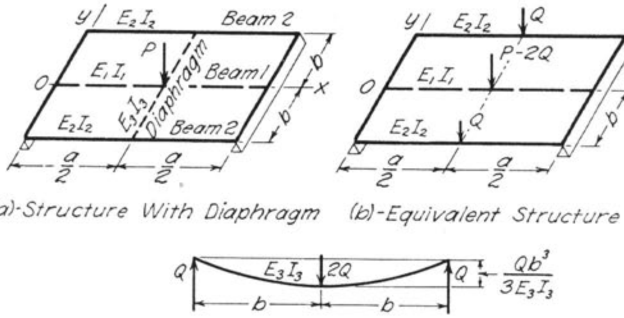
$$z\Big]_{x=a/3} = 0$$

gives the interior reaction

$$Q = \frac{pa^2}{2\pi^2} \frac{\displaystyle\sum_{1,3,5,\cdots} \frac{1}{n^5} d_3 \sin \frac{n\pi}{3}}{\displaystyle\sum_{1,3,5,\cdots} \frac{1}{n^3} f_7 \sin^2 \frac{n\pi}{3}}. \quad (186)$$

The quantities $b_3$, $c_3$, $d_3$, $f_2$, $f_4$ and $f_7$ are given in Appendix B.

33. *Rectangular Slab Having Edge and Center Stringers and a Continuous Diaphragm on the Lateral Center Line; Concentrated Load at the Center of the Slab.*—The slab is shown in Fig. 34 (a). The dia-




(a)-Structure With Diaphragm (b)-Equivalent Structure

(c)-Diaphragm

FIG. 34

phragm is assumed to have no contact with the slab, to be simply supported at the outer stringers, and to be continuous across the center stringer. Since the connections of an actual diaphragm would not provide perfect continuity across the center stringer but would offer some restraint at the edge beams, these assumptions are somewhat compensating in their effect on load transmission to the outer beams.

Let $Q$ be the end reaction on the diaphragm as shown in Fig. 34 (c). The deflection of the center of the diaphragm relative to the ends is then $Qb^3/(3E_3I_3)$.

The total deflection of the slab for $y > 0$ may be obtained by combining solutions of the types given in Sections 25 and 26. The net load over the middle beam is $P - 2Q$ and the outer loads are each equal to $Q$ as shown in Fig. 34 (b). Then, from (155) and (164), the deflection of Beam 1 at $x = a/2$ is found to be

$$\left. z_1 \right]_{x=a/2} = \frac{2(P-2Q)a^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2 f_1}{n^4} + \frac{4Qa^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2 f_2}{n^4}$$
$$= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2 f_1}{n^4} + \frac{4Qa^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2}{n^4} (f_2 - f_1). \quad (187)$$

From (157) and (165) the deflection of Beam 2 at $x = a/2$ is

$$\left. z_2 \right]_{x=a/2} = \frac{2(P - 2Q)a^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2 f_2}{n^4} + \frac{2Qa^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2 f_6}{n^4}$$
$$= \frac{2Pa^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2 f_2}{n^4} + \frac{2Qa^3}{\pi^4 E_1 I_1} \sum_{1,3,5,\cdots} \frac{k_2}{n^4} (f_6 - 2f_2). \qquad (188)$$

The difference between (187) and (188) must be equal to the relative deflection of the diaphragm, $Qb^3/(3E_3I_3)$. From this requirement, one finds

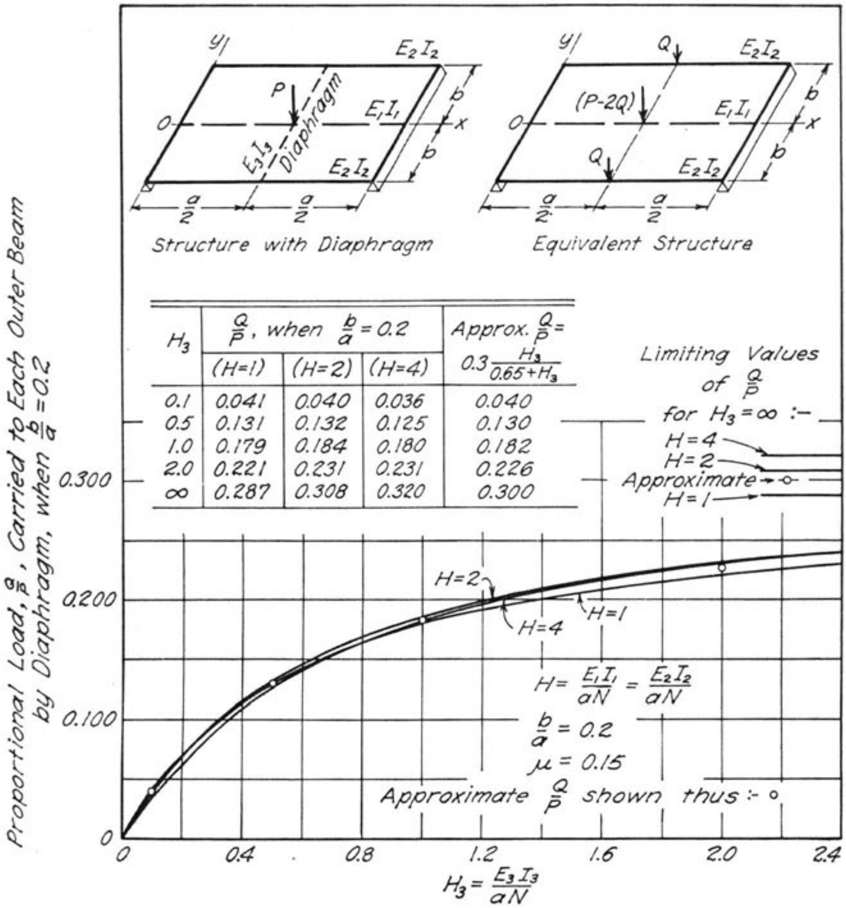
$$\frac{Q}{P} = \frac{\displaystyle\sum_{1,3,5,\cdots} \frac{k_2}{n^4}(f_1 - f_2)}{\displaystyle\frac{\pi^4}{6}\frac{E_1 I_1}{E_3 I_3}\frac{b^3}{a^3} + \sum_{1,3,5,\cdots} \frac{k_2}{n^4}(2f_1 + f_6 - 4f_2)} \qquad (189)$$

where

$$k_2 = \frac{1}{\dfrac{4}{n\pi H_1} + f_1}$$

and $f_1$, $f_2$ and $f_6$ are given in Appendix B.

As an example of the results given by (189), values of $Q/P$ were computed for successive values of $H = H_1 = H_2$ of 1.0, 2.0 and 4.0, taking $(b/a) = 0.2$ and letting $H_3$ assume various values. The results




| $H_3$ | $\frac{Q}{P}$, when $\frac{b}{a} = 0.2$ ($H=1$) | ($H=2$) | ($H=4$) | Approx. $\frac{Q}{P} = 0.3\frac{H_3}{0.65+H_3}$ |
|---|---|---|---|---|
| 0.1 | 0.041 | 0.040 | 0.036 | 0.040 |
| 0.5 | 0.131 | 0.132 | 0.125 | 0.130 |
| 1.0 | 0.179 | 0.184 | 0.180 | 0.182 |
| 2.0 | 0.221 | 0.231 | 0.231 | 0.226 |
| $\infty$ | 0.287 | 0.308 | 0.320 | 0.300 |

FIG. 35

are shown by the curves of Fig. 35. It will be noted that, for this ratio of $b/a$, the change in the value of $H$ has little effect within the range of values taken. As might be expected, the maximum ratio of $Q/P$, obtainable when the diaphragm becomes infinitely rigid, is slightly less than 1/3.

An equation of the form

$$\frac{Q}{P} = 0.3\frac{H_3}{0.65 + H_3} \tag{190}$$

represents the curves obtained from (189) for $b/a = 0.2$ and for $4.0 > H > 1.0$ as may be seen from the plotted results shown in Fig. 35.

The moments in the slab and beams may be obtained from the results of Sections 25 and 26, provided that the loads are assumed to be as shown in Fig. 34 (b) with $Q$ given by Equation (189).

34. *Rectangular Slab Having Edge and Center Stringers and a Continuous Diaphragm on the Lateral Center Line; Concentrated Load at the Center of an Edge Stringer.*—The slab with the actual loading is shown in Fig. 36 (a) and the slab without the diaphragm, but with

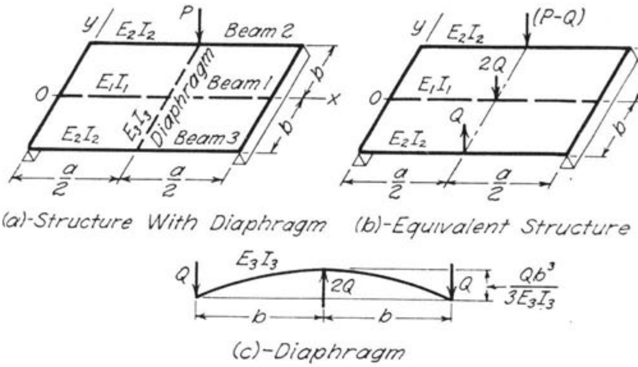


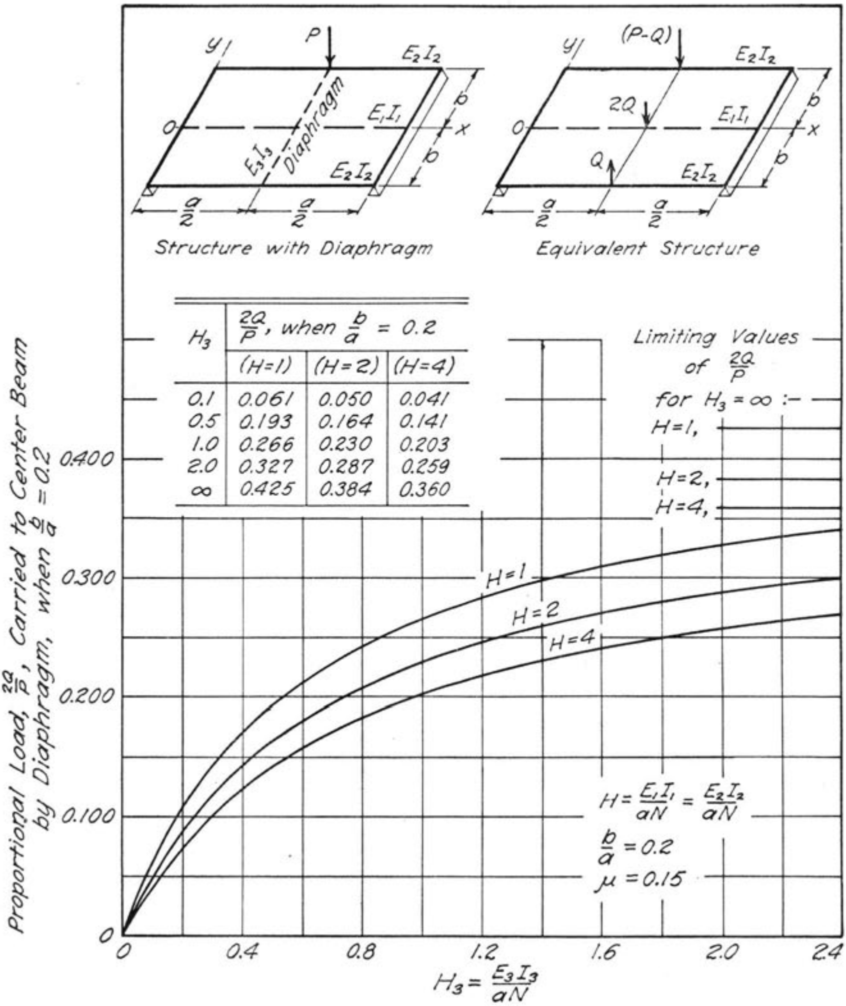

FIG. 36

an equivalent loading which includes the effect of the diaphragm, is shown in Fig. 36 (b). Under the same assumptions as are stated in Section 33, one may combine previously given solutions to obtain the ratio of the load carried over by the diaphragm, $2Q$, to the applied load, $P$. One finds

$$\frac{2Q}{P} = \frac{\sum\limits_{1,3,5,\cdots} \frac{k_2}{n^4} (f_6 - 2f_2)}{\frac{\pi^4}{6} \frac{E_1I_1}{E_3I_3} \frac{b^3}{a^3} + \sum\limits_{1,3,5,\cdots} \frac{k_2}{n^4} (2f_1 + f_6 - 4f_2)} \tag{191}$$

where the notation is the same as for Equation (189). Comparison between Equations (191) and (189) shows that they are identical except for one term in the numerator.

Figure 37 shows the ratio $2Q/P$ computed from (191) for $b/a = 0.2$,




| $H_3$ | $\frac{2Q}{P}$, when $\frac{b}{a} = 0.2$ (H=1) | (H=2) | (H=4) |
|---|---|---|---|
| 0.1 | 0.061 | 0.050 | 0.041 |
| 0.5 | 0.193 | 0.164 | 0.141 |
| 1.0 | 0.266 | 0.230 | 0.203 |
| 2.0 | 0.327 | 0.287 | 0.259 |
| ∞ | 0.425 | 0.384 | 0.360 |

FIG. 37

for successive values of $H_1 = H_2 = H$ of 1.0, 2.0 and 4.0, and for various values of $H_3$.

The conditions represented in Figs. 34 and 36 are extreme as to position of the load and the effect of the diaphragm. In one case the diaphragm is dished to its greatest extent, and in the other case it is humped its maximum amount. There is an intermediate position of the load on the slab for which the diaphragm will be straight and, according to the assumptions of the solution, will transmit no load. Its effect will then be to twist the stringers.

35. *Rectangular Slab, Having Two Opposite Edges Simply Supported and Two Edges Stiffened, Loaded by Symmetrical Concentrated Loads Placed on the Center Line Normal to the Stiffened Edges.*—The slab is shown in Fig. 38 where it will be noted that the axes have been

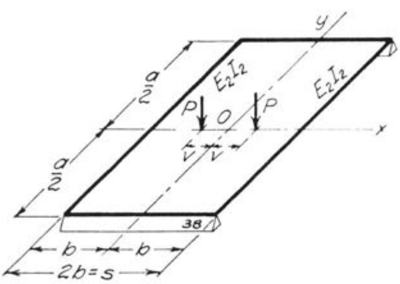



FIG. 38

rotated and shifted from the position given in the corresponding solution of Section 17. The purpose of the change is to put the solution into a form which, for small values of $b/a$, can utilize the well known solution for an infinitely long slab, having a span $2b = s$, simply supported on the edges $x = \pm b$, and supporting the given pair of concentrated loads. For all values of $b/a$ the solution may, however, be stated in a form which contains, in part, the solution of the rectangular slab, simply supported on four edges, and carrying the given loads.

Consider the portion of the slab between the limits $x = \pm v$. The deflection, $w_1$, is given by the sum of Equations (94) and (96) provided that proper correction is made for the position of the pair of loads and for the changed axes. One finds

$$\left.\begin{aligned} w_1 = \frac{Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \{ [b_1 + (1 + \alpha v)\, e^{-\alpha v}] \cosh \alpha x \\ - (c_1 + e^{-\alpha v})\, \alpha x \sinh \alpha x \} \cos \alpha y \end{aligned}\right\} \quad (192)$$

where $b_1$ and $c_1$ are given in Appendix B.

The corresponding solution, under the condition of simply supported edges on all four sides, is

$$w_0 = \frac{Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \left\{ \left[ b_1' + (1 + \alpha v) e^{-\alpha v} \right] \cosh \alpha x - (c_1' + e^{-\alpha v}) \alpha x \sinh \alpha x \right\} \cos \alpha y \qquad (193)$$

where

$$\left. \begin{aligned} b_1' &= \underset{H_2 \to \infty}{L}\, b_1 = (\tanh \alpha b - 1 - \alpha b \operatorname{sech}^2 \alpha b) \cosh \alpha v \\ &\qquad - (\tanh \alpha b - 1)\, \alpha v \sinh \alpha v, \\ c_1' &= \underset{H_2 \to \infty}{L}\, c_1 = (\tanh \alpha b - 1) \cosh \alpha v. \end{aligned} \right\} \qquad (194)$$

If one writes

$$w_1 = w_0 + w_{cor}, \qquad (195)$$

the corrective deflection function is

$$w_{cor} = \frac{Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \left[ (b_1 - b_1') \cosh \alpha x - (c_1 - c_1') \alpha x \sinh \alpha x \right] \cos \alpha y.$$

After reduction this becomes

$$w_{cor} = \frac{2Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{k_4}{n^3} \left[ a_4 \cosh \alpha x - (1 - \mu) \alpha x \sinh \alpha x \right] \cos \alpha y \qquad (196)$$

where

$$\left. \begin{aligned} a_4 &= 2 + (1 - \mu) \alpha b \tanh \alpha b, \\ \Delta k_4 &= a_4 \cosh \alpha v - (1 - \mu) \alpha v \sinh \alpha v, \end{aligned} \right\} \qquad (197)$$

and $\Delta$ is given in Appendix B. Since $w_{cor}$ is the effect of edge forces only, it is valid over the entire area of the slab.

When $b/a < 0.2$ one may take $w_0$ as the deflection of an infinitely long slab of span $2b = s$; that is, for $y > 0$,

$$w_0 = \frac{Ps^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \left( 1 + \frac{n\pi y}{s} \right) e^{-\frac{n\pi y}{s}} \cos \frac{n\pi v}{s} \cos \frac{n\pi x}{s}. \qquad (198)$$

The addition of Equation (196) to Equation (198) corrects for the flexibility of the beams at $x = \pm b$. In each of Equations (196) and (198) the apparent interchangeability of $x$ and $v$ demonstrates at once that these solutions maintain a reciprocal relationship between the deflection on lines $x = \pm x_1$ due to loads at $x = \pm v$ and the deflection on lines $x = \pm v$ due to loads at $x = \pm x_0$.

The deflection of either edge beam is

$$z = w\Big]_{x=b} = \frac{Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} f_5 \cos \alpha y. \tag{199}$$

36. *Rectangular Slab Having Two Opposite Edges Simply Supported and Two Edges Stiffened, Loaded by Anti-symmetrical Concentrated Loads Placed on the Center Line Normal to the Stiffened Edges.*—When the axes are placed as shown in Fig. 39 one may use a

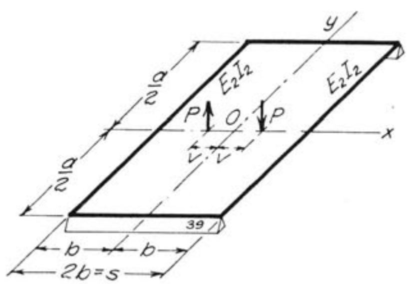



FIG. 39

procedure like that used in Section 35 to obtain a deflection function of the form

$$w = w_0 + w_{\text{cor}} \tag{200}$$

where $w_0$ is the deflection of a rectangular slab, $2b$ by $a$, simply supported on all edges, and loaded as shown. The corrective part of the solution, which accounts for the deflection of the edge beams, is found to be

$$w_{\text{cor}} = \frac{2Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{k_5}{n^3} \left[a_5 \sinh \alpha x - (1 - \mu)\, \alpha x \cosh \alpha x\right] \cos \alpha y \tag{201}$$

where

$$\left.\begin{aligned} a_5 &= 2 + (1 - \mu)\ \alpha b \coth \alpha b, \\ \Delta_1 k_5 &= a_5 \sinh \alpha v - (1 - \mu)\ \alpha v \cosh \alpha v, \end{aligned}\right\} \tag{202}$$

and $\Delta_1$ is given in Appendix B.

When $b/a < 0.4$ one may take $w_0$ as approximately equal to the deflection of an infinitely long slab, of span $2b = s$, supporting the given loads. Then, for $y > 0$,

$$w_0 = \frac{Ps^2}{\pi^3 N} \sum_{2,4,6,\cdots} \frac{1}{n^3} \left(1 + \frac{n\pi y}{s}\right) e^{-\frac{\pi y}{s}} \sin \frac{n\pi v}{s} \sin \frac{n\pi x}{s}. \tag{203}$$

The same reciprocal relationship mentioned in Section 35 may be seen to apply also to the slab loaded by the anti-symmetrical loads considered here.

The beam deflection at $x = b$ is

$$z = w\Big]_{x=b} = \frac{Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} F_5 \cos \alpha y. \tag{204}$$

37. *Rectangular Slab, Simply Supported on Four Edges, Having a Rigid Support Along the Longitudinal Center Line and Carrying a Concentrated Load on the Lateral Center Line.*—The axes may be

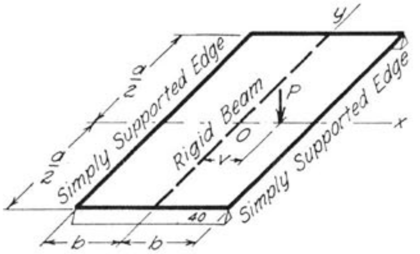



FIG. 40

oriented as shown in Fig. 40. A combination of the solutions given in Sections 20 and 24, when $u = a/2$ and $E_1 I_1 = E_2 I_2 = \infty$, then gives a deflection function

$$
\left.\begin{aligned}
w_1 = \frac{Pa^2}{2\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \{ & [a' + k_1' + (1 + \alpha v)\, e^{-\alpha v}] \sinh \alpha x \\
& - (c_1' - k_1' f_3' + e^{-\alpha v})\, \alpha x \sinh \alpha x \\
& + (d_1' - k_1' - e^{-\alpha v})\, \alpha x \cosh \alpha x \} \cos \alpha y
\end{aligned}\right\} \tag{205}
$$

which is valid when $0 < x < v$. In this equation the quantities having primes are each equal to the corresponding unprimed quantities evaluated at $E_1 I_1 = E_2 I_2 = \infty$. The unprimed quantities are stated in Appendix B.

If the slab is cut on the line $x = 0$, thus dividing it into two rectangular slabs simply supported on all edges, the deflection for $0 < x < v$ may be found from Section 20 to be

$$
\left.\begin{aligned}
w_1' = \frac{Pa^2}{\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \{ & [a_1' + (1 + \alpha v)\, e^{-\alpha v}] \sinh \alpha x \\
& + (d_1' - e^{-\alpha v})\, \alpha x \cosh \alpha x \} \cos \alpha y.
\end{aligned}\right\} \tag{206}
$$

The difference between (205) and (206) is, therefore, a corrective deflection function which provides for the continuity across the center beam.

One has

$$
\begin{aligned}
w_{\text{cor}} &= w_1 - w_1' \\
&= -\frac{Pa^2}{2\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \{ [a_1' - k_1' + (1 + \alpha v)\, e^{-\alpha v}] \sinh \alpha x \\
&\qquad + (c_1' - k_1' f_3' + e^{-\alpha v})\, \alpha x \sinh \alpha x \\
&\qquad + (d_1' + k_1' - e^{-\alpha v})\, \alpha x \cosh \alpha x \} \cos \alpha y.
\end{aligned}
$$

After substituting the values of $a_1'$, $c_1'$, $d_1'$, $k_1'$ and $f_3'$ into this equation, and reducing the resulting expression, one finds

$$
\left.\begin{aligned}
w_{\text{cor}} = -\frac{Pa^2}{2\pi^3 N} \sum_{1,3,5,\cdots} \frac{c_4}{n^3} & [(2\alpha b - \alpha x) \sinh \alpha x \\
& - \alpha x \sinh (2\alpha b - \alpha x)] \cos \alpha y
\end{aligned}\right\} \tag{207}
$$

where

$$c_4 = \frac{(2\alpha b - \alpha v)\sinh \alpha v - \alpha v \sinh (2\alpha b - \alpha v)}{(\sinh 2\alpha b - 2\alpha b)(\cosh 2\alpha b - 1)}. \tag{208}$$

Equation (207) is valid over the entire loaded panel. It also represents the total deflection in the unloaded panel provided that $x$ is taken positive to the left.

One has, therefore, the deflection of the slab, for $x > 0$,

$$w = w_0 + w_{\text{cor}}$$

where $w_0$ is the deflection of a rectangular slab, $b$ by $a$, simply supported on all edges and loaded as in the right-hand panel of Fig. 40, and $w_{\text{cor}}$ is given by Equation (207). When the ratio $b/a$ is small, that is when $(b/a) < 0.4$, one may take $w_0$ as approximately equal to the deflection of the infinitely long slab of span $b$. Then

$$w_0 = \frac{Pb^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3}\left(1 + \frac{n\pi y}{b}\right) e^{-\frac{n\pi y}{b}} \sin \frac{n\pi v}{b} \sin \frac{n\pi x}{b}. \tag{209}$$

The bending moment $M_x$ in the slab over the interior beam is

$$\left.\begin{aligned} M_x\Big]_{x=0} &= -N\frac{\partial^2 w}{\partial x^2}\Big]_{x=0} = -N\frac{\partial^2 w_{\text{cor}}}{\partial x^2}\Big]_{x=0} \\ &= \frac{Pv}{a}\sum_{1,3,5,\cdots} \frac{\left(\frac{2b}{v} - 1\right)\sinh \alpha v - \sinh (2\alpha b - \alpha v)}{\sinh 2\alpha b - 2\alpha b} \cos \alpha y. \end{aligned}\right\} \tag{210}$$

38. *The Infinitely Long Slab Simply Supported at the Edges, Continuous Over a Central Line Support, and Carrying a Concentrated Load.*—In the preceding solution, by allowing the supports at $y = \pm a/2$ to recede sufficiently far from the load, one may determine approximately the deflection of an infinitely long slab having a

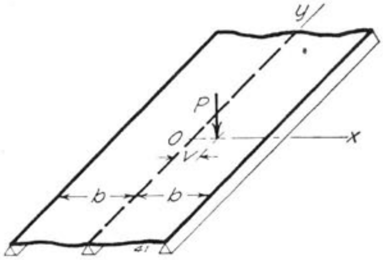



FIG. 41

central interior line support as shown in Fig. 41. The approximation is quite good if one lets $a = 2\pi b$ in Equations (207) and (208) and confines the application of the results to a region $0 < y < 2b$.

For convenience, one may use dimensionless variables:

$$\left.\begin{aligned} x_1 &= x/(2b), \\ y_1 &= y/(2b), \\ v_1 &= v/(2b). \end{aligned}\right\} \tag{211}$$

Then in the loaded strip, for $0 < y < 2b$, the deflection of the slab is

$$w = w_0 + w_{\text{cor}} \tag{212}$$

where

$$w_0 = \frac{Pb^2}{2\pi^3 N} \sum_{1,2,3,\cdots} \frac{1}{n^3} \left(1 + \frac{n\pi y}{b}\right) e^{-\frac{n\pi y}{b}} \sin \frac{n\pi v}{b} \sin \frac{n\pi x}{b}, \tag{213}$$

$$w_{\text{cor}} = -\frac{2Pb^2}{\pi N} \sum_{1,3,5,\cdots} \frac{d_4}{n} \left[(1-x_1) \sinh nx_1 - x_1 \sinh (n - nx_1)\right] \cos ny_1 \tag{214}$$

and

$$d_4 = \frac{(1 - v_1) \sinh nv_1 - v_1 \sinh (n - nv_1)}{(\sinh n - n)(\cosh n - 1)}. \tag{215}$$

The deflection of the unloaded panel, when $2b > y > -2b$, is given by (214) provided that $x$ is taken positive to the left.

The bending moment $M_x$ over the interior support, between the limits $2b > y > -2b$, is .

$$M_x\Big]_{x=0} = \frac{P}{\pi} \sum_{1,3,5,\cdots} \frac{(1-v_1)\sinh nv_1 - v_1 \sinh (n-nv_1)}{\sinh n - n} \cos ny_1. \quad (216)$$

39. *A Concentrated Load at the Center of a Rectangular Panel Having the Slab Continuous Over Two Opposite Edges and Simply Supported on the Other Edges.*—The slab is shown in Fig. 42 where

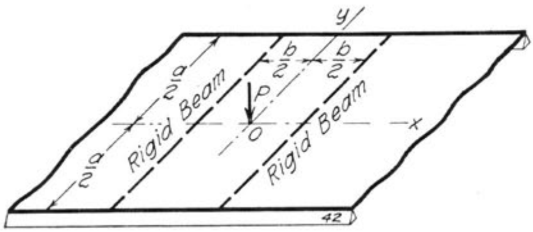



FIG. 42

it is noted that the origin of coördinates coincides with the loaded central point of the rectangular panel, the slab is non-deflecting at all supports, is continuous across the supports at $x = \pm b/2$, and is simply supported on the edges $y = \pm a/2$. Let the total deflection of the slab be denoted by $w_1$ for $0 < x < b/2$. An equation for this deflection may be obtained from Equations (47) and (48) provided that proper corrections are made for the changed positions of the load and axes. One finds

$$\left.\begin{aligned} w_1 = \frac{Pa^2}{2\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} \Big\{ &(1 + \alpha x)(\cosh \alpha x - \sinh \alpha x) \\ &- 2a_n \left[ \left(1 + \frac{\alpha b}{2}\right) \cosh \alpha x - \alpha x \sinh \alpha x \right] \Big\} \cos \alpha y \end{aligned}\right\} (217)$$

in which

$$a_n = \frac{1 + \dfrac{\alpha b}{2}}{1 + \alpha b + \sinh \alpha b + \cosh \alpha b}. \quad (218)$$

The solution represented by (217) is satisfactory to use when the ratio $b/a$ is not too small. When the ratio $b/a$ is small, in particular when it is so small that the effect of the supports at $y = \pm a/2$ becomes negligible, a bending moment obtained from (217) is in the form of a series whose convergence is very slow, and whose terms may be made up of small differences of large numbers. This difficulty may be avoided by putting $w_1$ in a special form in the following manner:

The deflection, $w_1$, may be written as the sum of two parts, $w_0$ and $w_{cor}$, where $w_0$ is the deflection of the rectangular slab, $b$ by $a$, simply supported on four edges, and loaded by a concentrated force $P$ at the center. The remaining part, $w_{cor}$, is a correction to be added to $w_0$ to give the effect of the continuity of the slab across the beams at $x = \pm b/2$. From Equation (109), with the modifications resulting from the position of the load and from the orientation of the axes, one finds

$$\left.\begin{aligned} w_0 = \frac{Pa^2}{2\pi^3 N} \sum_{1,3,5,\cdots} \frac{1}{n^3} (f_1' \cosh \alpha x - \sinh \alpha x \\ + \alpha x \cosh \alpha x - f_3' \alpha x \sinh \alpha x) \cos \alpha y \end{aligned}\right\} \tag{219}$$

where

$$f_1' = \frac{\sinh \alpha b - \alpha b}{1 + \cosh \alpha b}, \qquad f_3' = \frac{\sinh \alpha b}{1 + \cosh \alpha b}. \tag{220}$$

The corrective deflection function is found as the difference between $w_1$ and $w_0$. Thus the corrective deflection function is

$$\left.\begin{aligned} w_{cor} &= w_1 - w_0 \\ &= -\frac{Pab}{2\pi^2 N} \sum_{1,3,5,\cdots} \frac{1}{n^2} k_5 \left( \frac{\alpha b}{2} \tanh \frac{\alpha b}{2} \cosh \alpha x - \alpha x \sinh \alpha x \right) \cos \alpha y \end{aligned}\right\} \tag{221}$$

where

$$k_5 = \frac{\tanh \dfrac{\alpha b}{2}}{1 + \alpha b + \sinh \alpha b + \cosh \alpha b}. \tag{222}$$

Equation (221), which gives the effect of continuity over the supports at $x = \pm b/2$, may be compared with the equation obtained by Westergaard* for the effect of fixing the edges at $x = \pm b/2$. Westergaard's equation for the change of deflection caused by fixing the two edges at $x = \pm b/2$, in the present notation, may be written in the form

$$\left.\begin{aligned} w'_{\text{cor}} = -\frac{Pab}{2\pi^2 N} \sum_{1,3,5,\cdots} \frac{1}{n^2} \frac{\tanh \dfrac{\alpha b}{2}}{\alpha b + \sinh \alpha b} \left( \frac{\alpha b}{2} \tanh \frac{\alpha b}{2} \cosh \alpha x \right. \\ \left. - \alpha x \sinh \alpha x \right) \cos \alpha y. \end{aligned}\right\} \quad (223)$$

The two results are found to be identical except for the presence of the quantity $(1 + \cosh \alpha b)$ in the denominator of the parameter $k_5$. That is, the substitution of

$$k'_5 = \frac{\tanh \dfrac{\alpha b}{2}}{\alpha b + \sinh \alpha b} \quad (224)$$

for $k_5$ in (221) changes the effect from that due to continuity with a slab of large extent in the $x$ direction to that due to fixing the slab against rotation and deflection on the lines $x = \pm b/2$.

The corrective moments under the load, obtained from (221), are

$$\left.\begin{aligned} M_x^{\text{cor}}\Big]_{x=y=0} &= -\frac{Pb}{2a} \sum_{1,3,5,\cdots} k_5 \left[ 2 - (1 - \mu) \frac{\alpha b}{2} \tanh \frac{\alpha b}{2} \right], \\ M_y^{\text{cor}}\Big]_{x=y=0} &= -\frac{Pb}{2a} \sum_{1,3,5,\cdots} k_5 \left[ 2\mu + (1 - \mu) \frac{\alpha b}{2} \tanh \frac{\alpha b}{2} \right], \end{aligned}\right\} \quad (225)$$

in which $k_5$ is given by (222).

As in Section 38, the edges at $y = \pm a/2$ may be moved sufficiently far from the load that their effect on the moments in the vicinity of the load practically disappears. For the initial deflection,

*H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Wheel Loads, Public Roads, V. 11, No. 1, March, 1930, p. 20; see Equation 101.

$w_0$, given by (219), there may then be substituted the deflection of the infinitely long slab of span $b$, simply supported along the edges $x = \pm b/2$, and carrying a concentrated load at the center of the span. The supports at $y = \pm a/2$ are moved sufficiently far from the load by making $a = 2\pi b$ in Equations (221) and (222).

Let $M_{0x}$ and $M_{0y}$ designate the bending moments under the load when it is at the center of the span of an infinitely long slab having simply supported edges at $x = \pm b/2$. Let $M'_{0x}$ and $M'_{0y}$ represent similar moments when the edges of the slab at $x = \pm b/2$ are fixed. Let $M''_{0x}$ and $M''_{0y}$ represent similar moments when the slab extends a considerable distance in the directions of positive and negative $x$ and is continuous over non-deflecting supports at $x = \pm b/2$. Under these conditions, for Poisson's ratio $\mu = 0.15$, the moments under the load are

$$\text{with simply supported edges} \quad \begin{cases} M_{0x} = M_{0x} \\ M_{0y} = M_{0x} - 0.0676P \end{cases} \tag{226}$$

$$\text{with fixed edges} \quad \begin{cases} M'_{0x} = M_{0x} - 0.0699P \\ M'_{0y} = M_{0y} - 0.03863P = M_{0x} - 0.1063P \end{cases} \tag{227}$$

$$\text{with continuity of slab} \quad \begin{cases} M''_{0x} = M_{0x} - 0.0293P \\ M''_{0y} = M_{0y} - 0.01905P = M_{0x} - 0.0867P. \end{cases} \tag{228}$$

The numerical value of the corrections to $M_{0x}$ and $M_{0y}$ given in (228) were computed with $a = 2\pi b$ in the corresponding correction functions (225). The initial quantities required in (226) may be obtained from (18) or from the numerical values and curves given by Westergaard. The corrections to $M_{0x}$ and $M_{0y}$ noted in (227) were given by Westergaard* and were verified from (225) with $a = 2\pi b$ and with $k'_5$ substituted for $k_5$.

It may be observed from (227) and (228) that the correction to $M_{0x}$ due to continuity of the slab is 42 per cent of the correction due to fixing the slab. Similarly it may be observed that the correction to $M_{0y}$ due to continuity is 49 per cent of the correction due to fixing the slab. It is apparent, therefore, that the continuous slab is not, in this instance, one having a definite "percentage of fixity," but may, nevertheless, be treated as "50 per cent fixed" with a satisfactory

*H. M. Westergaard, Computation of Stresses in Bridge Slabs Due to Concentrated Loads, Public Roads, V. 11, No. 1, March, 1930, p. 20; see equations 104 and 105.

degree of approximation. The magnitude of such an approximation may be illustrated by an example in which the moment $M_{0x}$ is assumed to be $0.300P$. The resultant moment $M''_{0x}$ is found from Equation (228) to be

$$M''_{0x} = 0.300P - 0.029P = 0.271P$$

whereas an assumption that the edges are "50 per cent fixed" gives a moment

$$\text{approx. } M''_{0x} = 0.300P - 0.035P = 0.265P.$$

In this example the error in the resultant moment $M''_{0x}$ is only about 2 per cent.

## VIII. Discussion

40. *Concluding Remarks.*—This bulletin presents the formal solutions of a number of problems involving continuous slabs which support concentrated loads. The bulletin is intended primarily as a source of reference from which material may be drawn for application to such related problems as are selected for further study. No attempt has been made here to develop a design procedure. Solutions have been determined which, it is believed, are capable of furnishing a better understanding of the behavior of bridge slabs and their supporting beams.

Mathematical solutions, such as those contained herein, are subject to certain limitations. These limitations are a direct result of the assumptions made in the analysis. While it is recognized that such assumptions as homogeneity and perfect elasticity of material cannot be realized in an actual slab, the results obtained on the basis of these assumptions may be used as a fairly reliable measure of the behavior of the slab.

Formulas are given for the bending moments in the supporting beams as well as in the slabs. The magnitudes of the errors which may arise from the assumption that the supporting beams exert only vertical forces upon the slab are yet to be determined.

In a given problem a solution in the form of an infinite series is a suitable tool for analysis provided that the labor involved in its use can be justified. It is possible in some cases to observe the variables which enter a given problem and to select the form of solution which will be simplest to use. In general, a solution may be made to consist of two parts, a basic solution and a correction. The basic solution may be chosen in a variety of ways. The best choice is that one which makes the correction a small part of the total effect. This is illustrated by the examples discussed in Sections 35 to 39 inclusive. In any of these examples the factors which govern the choice of solution are: ratio of sides of panel, and relative rigidity of supporting beams. The solutions obtained in Sections 35 to 39 are convenient to use when the loaded panel is narrow and the supporting beams are relatively stiff.

Concerning the complexity of the solutions given, it is obvious that an equation for the deflection of a slab cannot be made to account for a variety of factors, such as relative stiffnesses of beams, ratio of sides of panel, and position of load, without serious complication. In other words, the effects of certain elements can be determined only at the expense of a considerable amount of labor. However, once these effects have been ascertained and verified, it becomes possible to judge and to improve, if necessary, existing empirical practice and, probably most important of all, to establish limits within which a given factor can affect a solution.

# APPENDIX A

## Use of the "Sign-function"

In the equations applicable to slabs it is frequently necessary to change the sign of certain terms when crossing specified rectilinear boundaries. Because of this fact it is expedient, in particular problems, to introduce a function which automatically provides the required change of sign when the boundary is crossed. The function is defined here and a few of its properties are stated.

By definition, let

$$\operatorname{sgn} y = \begin{cases} +1 & \text{for} \quad y > 0 \\ -1 & \text{for} \quad y < 0. \end{cases}$$

This function, interpreted as "sign of $y$," is mentioned by Rothe* in his discussion of the "impulse function" or "unit-function." Rothe assigns an arbitrary single value to sgn $y$ at $y = 0$. In applying the sign function to beams and slabs it is more appropriate to keep the double value of sgn 0, observing that, as $y$ approaches the origin from the positive side, one need only evaluate

$$\operatorname{sgn} y \Big]_{y=\epsilon} = +1$$

where $\epsilon$ is an infinitesimally small positive quantity. Similarly, very near to the origin on the negative side, one may use

$$\operatorname{sgn} y \Big]_{y=-\epsilon} = -1.$$

For the purpose of differentiation one may assign a zero value at $y = 0$ to the derivative of sgn $y$ with respect to $y$, so that

$$\frac{d}{dy} \operatorname{sgn} y \equiv 0.$$

*R. Rothe, F. Ollendorff and K. Polhausen, Theory of Functions as Applied to Engineering Problems, Translation by Alfred Hertzenberg, Technology Press, Massachusetts Institute of Technology, Cambridge, 1933, p. 61.

Certain other properties of sgn $y$ are important, namely,

$$\operatorname{sgn}^2 y \equiv 1$$

and

$$\operatorname{sgn}(-y) \equiv -\operatorname{sgn} y.$$

In the analysis of slabs the relationship "$y$ sgn $y$" frequently occurs. While this can always be represented by the symbol $|y|$, there is no provision in the latter representation for taking the derivative. It is sufficient, however, to write

$$|y| = y \operatorname{sgn} y$$

from which one obtains

$$\frac{d}{dy}|y| = \frac{d}{dy}(y \operatorname{sgn} y) = \operatorname{sgn} y.$$

A movement of the origin which results from the substitution of $(y - v)$ for $y$, where $v$ is a constant, has the effect, of course, of shifting the sign change from the line $y = 0$ to the line $y = v$.

In a number of the problems in the text use is made of the relationships just given. For example, in Equation (9) there appears a function of $y$, namely,

$$f(y) = (1 + \alpha|y-v|)\, e^{-\alpha|y-v|}$$

which may be written in the form

$$f(y) = [1 + \alpha (y - v) \operatorname{sgn}(y - v)]\, e^{-\alpha(y-v) \operatorname{sgn}(y-v)}.$$

Derivatives of this function with respect to $y$, when $\alpha$ and $v$ are constants, are

$$\begin{aligned}\frac{df}{dy} &= -\alpha^2 (y - v)\,[\operatorname{sgn}^2 (y - v)]\, e^{-\alpha(y-v)\operatorname{sgn}(y-v)}\\ &= -\alpha^2 (y - v)\, e^{-\alpha|y-v|},\end{aligned}$$

$$\begin{aligned}\frac{d^2f}{dy^2} &= -\alpha^2 [1 - \alpha (y - v) \operatorname{sgn}(y - v)]\, e^{-\alpha(y-v)\operatorname{sgn}(y-v)}\\ &= -\alpha^2 (1 - \alpha|y-v|)\, e^{-\alpha|y-v|},\end{aligned}$$

etc.

## APPENDIX B

### Summary of Formulas for Parametric Quantities

Certain quantities have been abbreviated in the text for convenience of expression. These quantities are listed below for easy reference.

$$\alpha = \frac{n\pi}{a}$$

$$\beta = \alpha b = \frac{n\pi b}{a}$$

$$N = \frac{Eh^3}{12\,(1 - \mu^2)}$$

$$H_1 = \frac{E_1 I_1}{aN}; \qquad H_2 = \frac{E_2 I_2}{aN}; \quad \text{etc.}$$

$$\Delta = 2n\pi H_2\,(1 + \cosh 2\beta) + (3 + \mu)\,(1 - \mu) \sinh 2\beta - 2\,(1 - \mu)^2\beta$$

$$\begin{aligned}\Delta_1 &= 2n\pi H_2\,(\cosh 2\beta - 1) + (3 + \mu)\,(1 - \mu) \sinh 2\beta + 2\,(1 - \mu)^2\beta \\ &= \Delta - 4n\pi H_2 + 4\,(1 - \mu)^2\beta\end{aligned}$$

$$\begin{aligned}\Delta f_1 = {} & 2n\pi H_2\,(\sinh 2\beta - 2\beta) + 4\,(1 + \mu) + (1 - \mu)^2\,(1 + 2\beta^2) \\ & + (3 + \mu)\,(1 - \mu) \cosh 2\beta\end{aligned}$$

$$\Delta f_2 = 4\,[2 \cosh \beta + (1 - \mu)\,\beta \sinh \beta]$$

$$\Delta f_3 = 2n\pi H_2 \sinh 2\beta + (1 - \mu)^2 + (3 + \mu)\,(1 - \mu) \cosh 2\beta$$

$$\Delta f_4 = 4\,(1 - \mu) \cosh \beta$$

$$\begin{aligned}\Delta f_5 = 4\,[ & 2 \cosh \beta \cosh \alpha v \\ & + (1 - \mu)\,(\beta \sinh \beta \cosh \alpha v - \alpha v \sinh \alpha v \cosh \beta)]\end{aligned}$$

$$\Delta f_6 = 2n\pi H_1 (\sinh 2\beta - 2\beta) + 8 (1 + \cosh 2\beta)$$

$$\Delta f_7 = 4 (1 + \cosh 2\beta)$$

$$\Delta f_8 = (3 + \mu) (1 - \mu) \sinh 2\beta - 2 (1 - \mu)^2 \beta$$

$$\Delta f_9 = n\pi H_1 [(1 + \mu) \sinh \beta + (1 - \mu) \beta \cosh \beta] - 4 (1 - \mu) \cosh \beta$$

$$\begin{aligned} \Delta_1 F_1 = {} & 2n\pi H_2 (\sinh 2\beta + 2\beta) - 4 (1 + \mu) - (1 - \mu)^2 (1 + 2\beta^2) \\ & + (3 + \mu) (1 - \mu) \cosh 2\beta \end{aligned}$$

$$\Delta_1 F_2 = 4 [2 \sinh \beta + (1 - \mu) \beta \cosh \beta]$$

$$\Delta_1 F_3 = 2n\pi H_2 \sinh 2\beta - (1 - \mu)^2 + (3 + \mu) (1 - \mu) \cosh 2\beta$$

$$\Delta_1 F_4 = 4 (1 - \mu) \sinh \beta$$

$$\begin{aligned} \Delta_1 F_5 = {} & 4 [2 \sinh \beta \sinh \alpha v \\ & + (1 - \mu) (\beta \cosh \beta \sinh \alpha v - \alpha v \cosh \alpha v \sinh \beta)] \end{aligned}$$

$$\Delta_1 F_7 = 4 (\cosh 2\beta - 1)$$

$$\Delta_1 F_8 = (3 + \mu) (1 - \mu) \sinh 2\beta + 2 (1 - \mu)^2 \beta$$

$$a_1 = (F_3 - 1) \alpha v \cosh \alpha v - (F_1 - 1) \sinh \alpha v$$

$$b_1 = (f_1 - 1) \cosh \alpha v - (f_3 - 1) \alpha v \sinh \alpha v$$

$$c_1 = (f_3 - 1) \cosh \alpha v - \frac{2 (1 - \mu)^2}{\Delta} \alpha v \sinh \alpha v$$

$$d_1 = (F_3 - 1) \sinh \alpha v + \frac{2 (1 - \mu)^2}{\Delta_1} \alpha v \cosh \alpha v$$

$$\Delta b_3 = 2n\pi H_2 (2 \cosh \beta + \beta \sinh \beta) + 2\mu \left[(1 - \mu) \beta \cosh \beta - (1 + \mu) \sinh \beta\right]$$

$$\Delta c_3 = 2n\pi H_2 \cosh \beta + 2\mu (1 - \mu) \sinh \beta$$

$$\Delta d_3 = (3 - \mu) \sinh 2\beta - 2 (1 - \mu) \beta$$

$$k_1 = \frac{b_1 + (1 + \alpha v) e^{-\alpha v}}{\dfrac{4}{n\pi H_1} + f_1}$$

$$k_2 = \frac{1}{\dfrac{4}{n\pi H_1} + f_1}$$

$$k_3 = \frac{1 - b_3}{\dfrac{4}{n\pi H_1} + f_1}$$

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